Further properties of $\mathbb{C}[[x]]$ (continued):

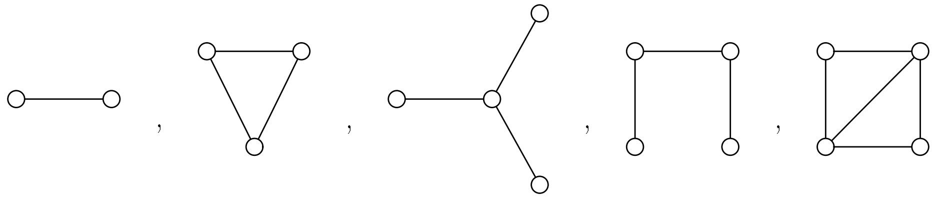
Composition: $A(B(x))$ defined if $B(x)$ has zero constant term, or if $A(x)$ is a polynomial.

Coefficient operator: The coefficient a_k of x^k in $A(x)$ will be denoted $[x^k]A(x)$. $[x^k]$ is a linear function on $\mathbb{C}[[x]]$.

Example:

$$[x^{12}]\,(1+2x+3x^2+4x^3+5x^4+\ldots) = 13$$
$$[x^k]\,1/(1-x) = [x^k]\,(1+x+x^2+x^3+\ldots) = 1$$

Multivariable generating functions: in combinatorial context, more than one weight function, use different variable for each weight function. E.g., plane graphs (drawn in plane without crossings), three weights: number of vertices, edges, faces (counting outside), marked by *indicator variables* x, y, z, respectively.

$A = \Big\{$  $\Big\}$

$$A(x,y,z) = x^2yz + x^3y^3z^2 + 2x^4y^3z + x^4y^5z^3.$$

Note that we can extract coefficients from multivariable g.f.s.

$$[x^4y^3z]\;A(x,y) = 2$$
$$[y^3]\;A(x,y) = x^3z^2 + 2x^4z$$
$$[x^0y^3z^0]\;A(x,y) = 0 \qquad \text{(not same thing as } [y^3])$$

Will mostly stick to single-variable g.f.s in this course.

Usefulness of g.f.s:

(1) Package whole sequence into one object.

(2) Algebraic manipulations of g.f. $\leftrightarrow$ common operations on sequences (or on sets we are counting).

Mult. by $x \leftrightarrow$ shift;

Differentiate $\leftrightarrow$ mult. a_k by k (and shift)

Mult. two g.f.s $\leftrightarrow$ convolution of sequences, cartesian product of sets we are counting.

(3) Switching to analytic perspective (think of x as number, worry about convergence) useful for working out asymptotic properties.

(4) If function can be expanded as a power series in x, often identify it with corresponding f.p.s. Behaviour of f.p.s. parallels behaviour of function.

E.g. $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$ (Maclaurin series) so use e^x to refer to this *formal* power series.

General approach:

(1) Set up a g.f. for set of objects to count.

(2) Build up g.f. from simpler g.f.s (Topic 2), or use a recurrence relation to find an equation for the g.f. (Topic 3).

(3) Extract counting information (coefficients) from actual g.f. or using tools to find coefficients based on an equation.

Before we go any further, need to look at commonly occurring g.f.s and how to extract coefficients from them.

Generalised Binomial Theorem for Formal Power Series: Let a be any rational number (can be written as a fraction), then

$$\begin{aligned}(1+x)^a &= 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots \\ &= \binom{a}{0} + \binom{a}{1}x + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \dots\end{aligned}$$

where we define the *generalised binomial coefficient*

$$\binom{a}{k} = \frac{a(a-1)(a-2)\dots(a-k+1)}{k!}$$

for any complex number a and any nonnegative integer k.

Notes: (1) This is actually the Maclaurin series for the function $(1+x)^a$.

(2) This equation is true in a formal sense: for example, $P(x) = (1+x)^{2/3}$ is the power series so that $P(x)^3 = (1+x)^2$.

Special cases: (1) $a = n$ is nonnegative integer: just get Corollary to ordinary Binomial Theorem.

(2) Suppose $a = -n$.

$$\begin{aligned}\binom{-n}{k} &= \frac{-n(-n-1)(-n-2)\dots(-n-k+1)}{k!} \\ &= (-1)^k \frac{(n+k-1)(n+k-2)\dots(n+1)n}{k!} = (-1)^k\binom{n+k-1}{k} = (-1)^k\left(\!\binom{n}{k}\!\right)\end{aligned}$$

and so we get

$$\begin{aligned}\frac{1}{(1+x)^n} &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}(-1)^k x^k = \sum_{k=0}^{\infty}\left(\!\binom{n}{k}\!\right)(-1)^k x^k \\ \frac{1}{(1-x)^n} &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k = \sum_{k=0}^{\infty}\left(\!\binom{n}{k}\!\right)x^k\end{aligned}$$

giving

$$* * * \qquad [x^k]\,\frac{1}{(1-x)^n} = \binom{n+k-1}{k} = \left(\!\binom{n}{k}\!\right) \qquad * * * \text{ MEMORISE!}$$

Did not assume anything about n, but frequently use when n is a positive integer, a a negative integer. We have seen numbers like this before.

(3) If we let $a = -1$, i.e. we let $n = 1$ in (2), we get

$$\frac{1}{1-x} = (1-x)^{-1} = \sum_{k=0}^{\infty}\binom{k}{k}x^k = 1 + x + x^2 + x^3 + \dots$$

which we have already seen.

Examples: (1) $[x^3]\,(1-x)^{-5} = \left(\!\binom{5}{3}\!\right) = \binom{7}{3} = 7.6.5/6 = 35.$

$$\begin{aligned}(2)\quad [x^k]\,\frac{1}{(1-2x)^5} &= [x^k]\sum_{k=0}^{\infty}\left(\!\binom{5}{k}\!\right)(2x)^k = [x^k]\sum_{k=0}^{\infty}\binom{k+4}{k}2^k x^k \\ &= \binom{k+4}{k}2^k = \binom{k+4}{4}2^k\end{aligned}$$

(3) $[x^r]\ (1-x^3)^{-6} = [x^r]\ \sum_{k=0}^{\infty} \left(\!\!\binom{6}{k}\!\!\right)(x^3)^k = [x^r]\ \sum_{k=0}^{\infty} \binom{k+5}{k} x^{3k}$

$$= \begin{cases} 0 & r \text{ not divisible by } 3 \\ \binom{k+5}{k} = \binom{k+5}{5} = \binom{r/3+5}{5} & r = 3k \end{cases}$$

(4) Find the g.f. for $\mathbb{N} = \{1, 2, 3, \ldots\}$ with $w(n) = n$.

$$\begin{aligned} \mathbb{N}(x) &= x + x^2 + x^3 + x^4 + \ldots = x(1 + x + x^2 + x^3 + \ldots) \\ &= x(1-x)^{-1} = \frac{x}{1-x} \end{aligned}$$

(5) Find g.f. for $A = \{$nonnegative integers divisible by 3$\}$ weighted by value (i.e. $w(n) = n$ again).

$$\begin{aligned} A(x) &= 1 + x^3 + x^6 + x^9 + \ldots = 1 + x^3 + (x^3)^2 + (x^3)^3 + \ldots \\ &= (1-x^3)^{-1} = \frac{1}{1-x^3} \end{aligned}$$

(6) $[x^9]\ x^3(1-x)^{-4} = [x^6]\ (1-x)^{-4} = \left(\!\!\binom{4}{6}\!\!\right) = \binom{9}{6} = \binom{9}{3} = \frac{9.8.7}{6} = 84$

In general: $[x^k]\ x^m A(x) = \begin{cases} [x^{k-m}]\ A(x) & k \geq m \\ 0 & k < m \end{cases}$

Now want to look at how we can combine simple g.f.s to get g.f.s of more complicated sets.

Sum Lemma: Suppose A and B are disjoint subsets of some set S with weight function w, and $C = A \cup B$. Then $C(x) = A(x) + B(x)$. (Same condition as Sum Rule in Topic 1.)

Example: Subsets of $\mathbb{N}_n = \{1, 2, ..., n\}$, weight = cardinality.

$A = \{$even subsets$\}$ $\qquad A(x) = \binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \ldots$

$B = \{$odd subsets$\}$ $\qquad B(x) = \binom{n}{1}x + \binom{n}{3}x^3 + \binom{n}{5}x^5 + \ldots$

$C = A \cup B = \{$all subsets$\}$ $\qquad C(x) = \binom{n}{0}x + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots = A(x) + B(x)$

Products of generating functions

Example: (1) Subsets of $\mathbb{N}_0$, with weight = value: $A = \{3, 4\}$, $A(x) = x^3 + x^4$; $B = \{2, 7, 9\}$, $B(x) = x^2 + x^7 + x^9$. Then $A \times B = \{(3, 2), (3, 7), (3, 9), (4, 2), (4, 7), (4, 9)\}$.
Weight elements of $A \times B$ by $w(a, b) = a + b$. Then

$$\begin{aligned} (A \times B)(x) &= x^5 + x^{10} + x^{12} + x^6 + x^{11} + x^{13} \\ &= x^{3+2} + x^{3+7} + x^{3+9} + x^{4+2} + x^{4+7} + x^{4+9} \\ &= (x^3 + x^4)(x^2 + x^7 + x^9) = A(x)B(x) \end{aligned}$$

Product Lemma: Suppose have A weighted by u, B weighted by v, and we weight $A \times B$ by $w(\alpha, \beta) = u(\alpha) + v(\beta)$. Then

$$\begin{aligned} (A \times B)_w(x) &= \sum_{(\alpha,\beta)\in A\times B} x^{w(\gamma)} = \sum_{(\alpha,\beta)\in A\times B} x^{u(\alpha)+v(\beta)} = \sum_{(\alpha,\beta)\in A\times B} x^{u(\alpha)}x^{v(\beta)} \\ &= \sum_{\alpha\in A}\sum_{\beta\in B} x^{u(\alpha)}x^{v(\beta)} = \sum_{\alpha\in A} x^{u(\alpha)} \sum_{\beta\in B} x^{v(\beta)} = A_u(x)B_v(x) \end{aligned}$$

Note: Extends to arbitrary cartesian products $A_1 \times A_2 \times \ldots \times A_k$.

Bijection Rule: Suppose have S weighted by w, S' weighted by w' and bijection $f : S \to S'$ with $w'(f(\sigma)) = w(\sigma)$ (weight-preserving) for all $\sigma \in S$. Then $S_w(x) = S'_{w'}(x)$.

Combine to get:
General Product Lemma: Suppose have A weighted by u, B weighted by v, injective $(1-1)$ operation $* : A \times B \to C$ and $R = \{\alpha * \beta \mid \alpha \in A, \beta \in B\}$ (range of $*$) weighted by $w(\alpha * \beta) = u(\alpha) + v(\beta)$. Then $R_w(x) = A_u(x)B_v(x)$. Since $*$ is injective, provides weight-preserving bijection from $A \times B$ to R.