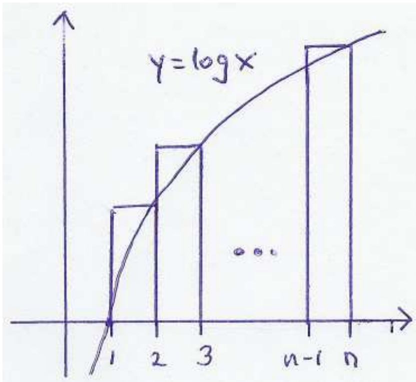

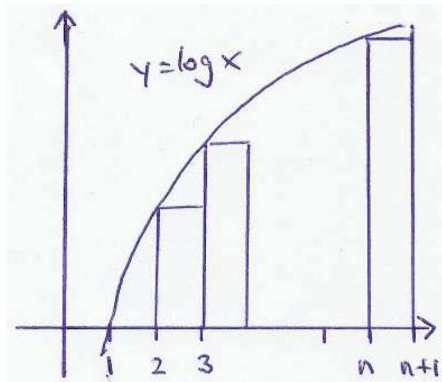

$(\log x = \log_e x = \ln x)$

Idea of why Stirling's formula is true:

$$\begin{aligned} n! &= 1.2.3.\ldots.n \\ \ln n! &= \ln 1 + \ln 2 + \ln 3 + \ldots + \ln n \\ &= \ln 2 + \ln 3 + \ldots + \ln n \end{aligned}$$

from which we see that

$$\ln n! \geq \int_1^n \ln x \, dx = [x \ln x - x]_1^n = n \ln n - n + 1$$

and

$$\ln n! \leq \int_2^{n+1} \ln x \, dx = [x \ln x - x]_2^{n+1} = (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2$$

from which we see that

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{(n+1)^{n+1}}{4e^{n-1}}.$$

By using techniques from advanced calculus can prove Stirling's formula or more precise version.

Example: Find an asymptotic estimate for the number of ways to split $2n$ people into red and blue teams of equal size:

Solution:

$$\begin{aligned} \binom{2n}{n} &= \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \\ &\sim \frac{\sqrt{4\pi n}(2n/e)^{2n}}{(\sqrt{2\pi n}(n/e)^n)^2} \\ &= \frac{2\sqrt{\pi n}4^n n^{2n} e^{-2n}}{2\pi n.n^{2n}e^{-2n}} \\ &= \frac{4^n}{\sqrt{\pi n}} \qquad \text{as } n \to \infty. \end{aligned}$$

So behaviour exponential, grows a little slower than 4^n.

2. GENERATING FUNCTIONS

Idea: represent counting information as power series and manipulate in ways like manipulate ordinary functions.

Combinatorial ordinary generating functions

Examples: (1) $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, the set of nonnegative integers, weight = value.

$\mathbb{N}_0(x) = 1 + x + x^2 + x^3 + \ldots = 1/(1-x)$ (will justify later).

(2) Take set of all 01-strings, weight = length.

$A = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$ (ε = "empty string" of length 0)
$A(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots = \sum_{k=0}^{\infty} 2^k x^k = 1/(1-2x)$

or

$B = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$
$B(x) = 1 + 2x + 4x^2$

Standard setup: Use generating functions when we have a collection of objects of different sizes, let a_0 be number of size 0, a_1 number of size 1, etc. Set up generating function, work out coefficient of x^k to get number of objects of size k.

A = set of objects
w = weight function (measure of "size" for elements of A: $w(\alpha)$ = size of α)
a_k = number of elements of size k (as measured by w) (must be finite for each k)

The *generating function of A weighted by w* is then

$$A_w(x) = A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k = \sum_{\alpha \in A} x^{w(\alpha)}$$

We drop the w subscript most of the time, since it is usually clear how we are measuring size. Two different ways to think of A: in terms of sequence $(a_0, \ldots)$ or by adding up a term for each element of set.

(3) Sets, weight = cardinality (no. of elements)

$\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$
$A = \mathcal{P}(\mathbb{N}_n) = \{\text{all subsets of } \mathbb{N}_n\}$ (*power set* of $\mathbb{N}_n$)
a_k = no. of k-subsets of $\mathbb{N}_n = \binom{n}{k}$
$A(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$ (Corollary to Binomial Theorem which works since multiply formal power series in same way as usual polynomials)

Not a coincidence that n-set gives $(1+x)^n$: will see why this happens later: get $(1+x)$ because each element in set (x) or out of set (1), and raise to power n because have n elements.

So $(1+x)^n$ is g.f. for subsets of $\mathbb{N}_n$ weighted by cardinality, i.e. g.f. for combinations selected from n objects, weighted by no. of objects chosen.

General ordinary generating functions

Will look at another sort of generating functions, exponential g.f.s, later.

Generating function of a sequence: The *(ordinary) generating function* (*o.g.f.* or *g.f.*) of a sequence $(a_0, a_1, a_2, \ldots) = (a_k)_{k=0}^{\infty}$ is

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

Belongs to $\mathbb{C}[[x]]$, the *ring of (complex) formal power series* (*f.p.s.*). Ring = set with +, × obeying sensible rules. This is commutative ring with identity. Also vector space over $\mathbb{C}$.

Formal power series means we are just to think of x as a symbol, not necessarily standing for an actual number. This means we aren't worried about convergence of power series or anything. Power series is just an algebraic object representing the sequence.

G.f.s appear in number of areas: e.g. probability. We are going to use them to represent counting information.

Example: (1) Sequence $(1, 2, -3, 0, 0, \dots)$ has g.f. $1 + 2x - 3x^2$. Polynomial – only finitely many nonzero terms.

(2) Sequence $(a_k)_{k=0}^{\infty} = (1, -1, 1, -1, \dots) = ((-1)^k)$ has g.f. $1 - x + x^2 - x^3 + x^4 - x^5 \dots$.

(3) Sequence $(a_k)_{k=0}^{\infty} = (1, 1, 2, 6, 24, \dots) = (k!)$ has g.f. $1 + x + 2x^2 + 6x^3 + 24x^4 + \dots$. Does not converge except at $x = 0$, but we don't care: perfectly good formal power series.

Operations on f.p.s.:

$$\begin{aligned}\text{Addition:}\quad & (a_0 + a_1x + a_2x^2 + \dots) + (b_0 + b_1x + b_2x^2 + \dots)\\ & \qquad = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots\\ \text{Multiplication:}\quad & (a_0 + a_1x + a_2x^2 + \dots) \times (b_0 + b_1x + b_2x^2 + \dots)\\ & \qquad = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots\end{aligned}$$

+ is closed, assoc., commut., identity, inverses; × is closed, assoc., commut., identity. × distributes over +.

Can also take negatives (additive inverse) and subtract.

Example:

$$\begin{aligned}A(x) &= 1 - x\\ B(x) &= 1 + x + x^2 + x^3 + x^4 + \dots\\ A(x) + B(x) &= 2 + x^2 + x^3 + x^4 + \dots\\ A(x) - B(x) &= -2x - x^2 - x^3 - x^4 - \dots\\ A(x)B(x) &= 1 + (-1+1)x + (-1+1)x^2 + (-1+1)x^3 + \dots = 1\end{aligned}$$

Since $A(x)B(x) = 1$ we say that $B(x)$ is the *reciprocal* (or mult. inverse) of $A(x)$ and write $B(x) = A(x)^{-1} = 1/A(x)$, i.e.

$$1 + x + x^2 + x^3 + x^4 + \dots = (1 - x)^{-1} = 1/(1 - x).$$

Further properties of $\mathbb{C}[[x]]$:

Cancellation: There are no zero divisors: if $A(x)B(x) = 0$ then either $A(x) = 0$ or $B(x) = 0$. Hence, if $A(x)B(x) = A(x)C(x)$ and $A(x) \neq 0$, then $B(x) = C(x)$. Even if $A(x)$ has no reciprocal.

Reciprocals/division: $A(x)$ has a reciprocal if and only if $a_0 \neq 0$. The reciprocal is unique. So we can always divide by any f.p.s with a nonzero constant term. Can also divide whenever numerator of same or higher degree than denominator by cancelling powers of x first.

Rational powers: Define using repeated multiplication, inverses, roots, e.g., $A(x)^5$, $A(x)^{-3}$, $A(x)^{1/4}$ (take pos. const. term), $A(x)^{-7/9}$.

Differentiation: $A'(x) = \sum_{k=1}^{\infty} ka_kx^{k-1}$ ($k = 0$ term is zero so omit) $= \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$.

Distance and Limits: Define *'norm'* $\|A(x)\|$ to be 0 if $A(x) = 0$, and 2^{-k} otherwise where a_k is the first nonzero coefficient. Not really a norm in vector space sense, hence quotes.

Then have *metric* (distance function; actually ultrametric, satisfies strong triangle inequality) $d(A(x), B(x)) = \|A(x) - B(x)\|$. Can define limits: $\lim_{n\to\infty} A_n(x) = A(x)$ means that $\lim_{n\to\infty} \|A_n(x) - A(x)\| = 0$. Means eventually agree with $A(x)$ up to any given term. E.g.

$$1, 1 + x, 1 + x + x^2, \ldots \to 1 + x + x^2 + x^3 + \ldots = 1/(1 - x).$$