Example: What is the probability of a flush in poker? (5-card, no drawing, count as a flush *every* hand with all cards of same suit).

Solution: Total number of hands is $\binom{52}{5}$.

To construct a flush: pick a suit (4 ways) then choose 5 cards in that suit ($\binom{13}{5}$ ways); total is $4\binom{13}{5}$ ways. So, probability is

$$4\binom{13}{5}\Big/\binom{52}{5} = \frac{4P(13,5)}{5!}\Big/\frac{P(52,5)}{5!} = \frac{4P(13,5)}{P(52,5)}$$
$$= \frac{4.13.12.11.10.9}{52.51.50.49.48} = \frac{11.3}{17.5.49.4} = \frac{33}{16\,660} \simeq 0.001\,98.$$

Can use combinations to solve 'arrangement with limited repetition' problems.

Example (revisited): Find number of arrangements of 3 red, 4 green, 5 blue balls.

New solution: Choose positions for red: $\binom{12}{3}$

Choose positions for green: $\binom{9}{4}$

Choose positions for blue: $\binom{5}{5}$

So total number of arrangements is

$$\binom{12}{3}\binom{9}{4}\binom{5}{5} = \frac{12!}{3!9!}\,\frac{9!}{4!5!}\,\frac{5!}{5!0!} = \frac{12!}{3!4!5!} = 27\,720$$

as before.

Can also look at combinations with repetition.

Example: Eight people go into a restaurant which serves three types of drink. Each person orders a drink. How many different drink orders are there *from the kitchen's point of view*? From waiter's point of view there are clearly 3^8.

Solution: Imagine waiter's order form:

Coke	Sprite	Orange
xx	xxx	xxx

Any order can be specified by eight x's and two bars: xx|xxx|xxx. Can choose position of bars in $\binom{10}{2} = \frac{10.9}{2} = 45$ ways.

In general: To select r objects from an infinite supply of n types of object, think of making an order form as above. We need r x's for the r objects, and $n-1$ bars to separate the n types of object; therefore the number of possible orders is

$$\left(\!\binom{n}{r}\!\right) \quad \text{"}n\text{ multichoose }r\text{"} \quad = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}.$$

Uses 'rising factorial' where $\binom{n}{r}$ uses 'falling factorial'. Interpretation: put down $n-1$ dividing bars first, then put r *distinct* x's: have n places to put first x, $n+1$ for second x, etc. (like counting ways to arrange r distinct books on n distinct shelves); but x's really identical so divide by $r!$.

Note: For integers n, r with $n \geq 0$: $\left(\!\!\binom{n}{0}\!\!\right) = 1$, $\left(\!\!\binom{0}{r}\!\!\right) = 0$ for $r \geq 1$, and by convention $\left(\!\!\binom{n}{r}\!\!\right) = 0$ for $r < 0$.

Distributions (usually equivalent to arrangement or selection with repetition)

First look at distributing distinct objects, e.g. people.

Example: In how many ways can we distribute six people into three rooms?

Solution: $3^6 = 729$, since 3 choices for each person.

In general: Number of ways to distribute r distinct objects into n (distinct) boxes is n^r.

Now look at distributing identical objects.

Example: How many ways are there to distribute 10 identical balls into 5 (distinct) boxes?

Solution: Once again use order form with 5 'compartments': each compartment corresponds to a box. So, we will need 10 x's for the balls and 4 bars to separate them into the 5 boxes. So answer is

$$\left(\!\!\binom{5}{10}\!\!\right) = \binom{14}{10} = \binom{14}{4} = \frac{14.13.12.11}{24} = 7.13.11 = 1\,001.$$

In general: Number of ways to distribute r identical objects into n distinct boxes: need an 'order form' with r x's for the objects, and $n-1$ bars to separate them into n boxes. So answer is $\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$.

Can also look at distributions of several types of object.

Example: In snooker there are 9 red balls and 6 other balls of different colours, A snooker table has 6 pockets. In how many ways can the balls be distributed to the pockets?

Solution: Distribute red balls $\left(\!\!\binom{6}{9}\!\!\right) = \binom{9+6-1}{6-1} = \binom{14}{5}$ ways

Distribute remaining balls 6^6 ways

The distributions are independent, so total of

$$6^6\binom{14}{5} = 6^6\frac{14.13.12.11.10}{5.4.3.2.1} = 6^6.14.13.11 = 6^6.2002 = 93\,405\,312.$$

Example: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 9$$

are there if all the x_i's are integers and

(a) all x_i's are nonnegative;
(b) all x_i's are positive;
(c) all x_i's are nonnegative and at least one is 0?

Note that $x_1 = 9$, $x_2 = x_3 = x_4 = x_5 = 0$ is to be thought of as a different solution from $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 9$: which variable gets which value matters.

Solution: Think of trying to distribute 9 balls into 5 boxes, where x_1 is the number of balls in box 1, x_2 is the number of balls in box 2, and so on.

(a) We have to distribute 9 balls into 5 boxes, so (need order form with 9 x's, 4 bars) number is

$$\left(\!\!\binom{5}{9}\!\!\right) = \binom{9+4}{4} = \binom{13}{4} = \frac{13.12.11.10}{4.3.2.1} = 13.11.5 = 715.$$

(b) Since each x_i must be ≥ 1, think of dropping one ball into each of the five boxes before we start. Then we are left with 4 balls to distribute into 5 boxes, which can be done in

$$\left(\!\binom{5}{4}\!\right) = \binom{4+4}{4} = \binom{8}{4} = \frac{8.7.6.5}{4.3.2.1} = 7.2.5 = 70$$

ways.

(c) = (a) − (b) = 715 − 70 = 645.

In general: The number of nonnegative integral solutions to

$$x_1 + x_2 + \ldots + x_n = r$$

is the number of ways of distributing r identical objects into n distinct boxes, i.e. the number of order forms with r x's and $n-1$ bars, i.e. $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$.

So combinations with repetitions, distributions of identical objects, and number of integral solutions of sum equations, are all same problem in some sense.

Example: I go into a candy shop with \$2. Bubblegum, mints and lemon drops are 10c each, while chocolate bars are 50c. If I want to spend all of my money, how many different combinations of these four types of candy can I get?

Solution: We want the number of nonnegative integral solutions to

$$10B + 10M + 10L + 50C = 200$$
$$\text{or} \quad B + M + L + 5C = 20$$

Divide up into cases according to the value of C:

$C = 0$	$B + M + L = 20$	$\left(\!\binom{3}{20}\!\right) = \binom{20+2}{2} = \binom{22}{2} = 231$ solutions
$C = 1$	$B + M + L = 15$	$\left(\!\binom{3}{15}\!\right) = \binom{15+2}{2} = \binom{17}{2} = 136$ solutions
$C = 2$	$B + M + L = 10$	$\left(\!\binom{3}{10}\!\right) = \binom{10+2}{2} = \binom{12}{2} = 66$ solutions
$C = 3$	$B + M + L = 5$	$\left(\!\binom{3}{5}\!\right) = \binom{5+2}{2} = \binom{7}{2} = 21$ solutions
$C = 4$	$B + M + L = 0$	$\left(\!\binom{3}{0}\!\right) = \binom{0+2}{2} = \binom{2}{2} = 1$ solution

so we get a total of $231 + 136 + 66 + 21 + 1 = 455$ different combinations.

Stanley's "12-fold way": Table for distributions of balls into boxes, can partly fill in.

r balls $\rightarrow$	n boxes	any way	≤ 1 ball per box	≥ 1 ball per box
distinct	distinct	n^r	$P(n, r)$	(from Stirling number)
identical	distinct	$\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$	$\binom{n}{r}$	$\left(\!\binom{n}{r-n}\!\right) = \binom{r-1}{r-n}$
distinct	identical	(from Stirling number)	1 if $r \leq n$ 0 if $r > n$	(Stirling number)
identical	identical	(part. r into $\leq n$ parts)	1 if $r \leq n$ 0 if $r > n$	(part. r into $= n$ parts)

Binomial coefficients

Consider the product

$$(x+ay)(x+by)(x+cy) = x^3 + (a+b+c)x^2y + (ab+ac+bc)xy^2 + abcy^3$$

The term $x^{3-k}y^k$ appears with all $\binom{3}{k}$ products of k of a, b and c. So, if $a=b=c=1$ we get

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3.$$

We can generalise this to products $(x+y)^n$.

Binomial Theorem: For any integer $n \geq 0$,

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.\end{aligned}$$

Binomial because we are looking at powers of sum of two variables.

Often want special case where first variable replaced by constant 1, second variable is x.

Corollary: For any integer $n \geq 0$,

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^3 + \binom{n}{n}x^n \\ &= 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + x^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k.\end{aligned}$$

We call $\binom{n}{k}$ a *binomial coefficient* because it turns up in the Binomial Theorem.

Properties: (1) $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \binom{n}{n-k}$ have already seen this.

(2) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof: To select k people from n, we have two choices:
– include nth person, choose $k-1$ from first $n-1$
– exclude nth person, choose all k from first $n-1$.
The result follows. ∎

We can use this to construct *Pascal's triangle.*

Pascal's triangle:

$n=0$						1					
$n=1$					1		1				
$n=2$				1		2		1			
$n=3$			1		3		3		1		
$n=4$		1		4		6		4		1	
$n=5$	1		5		10		10		5		1

This gives us $\binom{n}{k}$: as we go across a row k goes from 0 to n. To get any entry, add the two entries above it.

Can interpret Pascal's triangle in terms of walking blocks in a rectangular grid. See Tucker for more details.

Binomial identities: Equations involving relationships between binomial coefficients.

(1) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n.$

Proof 1: Use Corollary to Binomial Theorem, let $x = 1$: get

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1}1^1 + \binom{n}{2}1^2 + \ldots \binom{n}{n}1^n$$
$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}. \quad \blacksquare$$

Proof 2: Use combinatorial (counting) argument. Suppose we want to choose an arbitrary subset of an n-set. For each element of the set, there are 2 choices: it is in the subset or out of it. So, there are 2^n subsets.
On the other hand, the number of subsets of size k is just $\binom{n}{k}$, and k can be anything from 0 to n. So the number of subsets is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}.$$

The two ways of counting the number of subsets must give equal answers, proving the result. ∎

(2) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

Proof 1: Counting argument. Suppose have n balls, want to paint k of them, with m red, $k-m$ green. Can count possibilities two ways. First, choose k to paint from n, then m to be red from k, giving $\binom{n}{k}\binom{k}{m}$. Second, choose k to paint red from n then $k-m$ to paint green from $n-m$, giving $\binom{n}{m}\binom{n-m}{k-m}$. ∎

Proof 2: Using algebra:

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!}$$
$$= \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!(n-k)!} = \binom{n}{m}\binom{n-m}{k-m}. \quad \blacksquare$$

Special case of (2)**:** Let $m = 1$, get $\binom{n}{k}k = n\binom{n-1}{k-1}$, giving

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$