Math 4630/6630 - Nonlinear Optimization - Spring 2021

Questions for Topic 2

Recall that when solving a system of nonlinear equations you must show full working. You may only use computational tools to solve LINEAR systems of equations. If a linear system occurs as a subproblem you may use computational tools to solve it.

2A. The problem of minimizing $f(x) = 2x_1^2+x_2^2+2x_1x_2-4x_1-5x_2+x_3$ subject to $2x_1+x_2+x_3 = 0$ is known to have a solution. Use Lagrange multipliers to find it. You should deal with the issue of whether a constraint qualification holds. However, you do not need to prove that your answer is a minimizer (rather than a maximizer or saddle point).

2B. The problem of maximizing $f(x) = 6x + 2y^2 + z$ subject to $2z - 3y = 1$ and $2x + y^2 - z = 0$ is known to have a solution. Use Lagrange multipliers to find it. You should deal with the issue of whether a constraint qualification holds. However, you do not need to prove that your answer is a maximizer (rather than a minimizer or saddle point).

2C. The problem of minimizing $f(x) = x_2 + 1$ subject to $x_2^3 = x_1^4$ is known to have a unique global solution. Use Lagrange multipliers to find it. You should deal with the issue of whether a constraint qualification holds. However, you do not need to prove that your answer is a minimizer (rather than a maximizer or saddle point). [**Note: this question is not totally straightforward!**]

2D. The problem of minimizing $f(x) = x_1^2 - 16x_1 + 4x_2^2 - 48x_2$ subject to $x_1 + 2x_2 \leq 7$ is known to have a solution. Use the Karush-Kuhn-Tucker conditions to find it. You should deal with the issue of whether a constraint qualification holds. However, you do not need to prove that your answer is really a minimizer (rather than a maximizer or saddle point).

2E. The problem of minimizing $f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 4x_2$ subject to $3x_1 + x_2 \leq 13$ is known to have a solution. Use the Karush-Kuhn-Tucker conditions to find it. You should deal with the issue of whether a constraint qualification holds. However, you do not need to prove that your answer is really a minimizer (rather than a maximizer or saddle point).

2F. The Separating Hyperplane Lemma. A *closed* set in $\mathbf{R}^n$ is one that contains its boundary. A *cone* in $\mathbf{R}^n$ is a nonempty set C such that $\alpha c \in C$ whenever $c \in C$ and $\alpha \geq 0$.

Suppose C is a closed convex cone in $\mathbf{R}^n$, and suppose $x^* \notin C$. Because C is closed, there is a point $c^* \in C$ that is a closest point in C to x^* (this is a general property of closed sets, and where we use the fact that C is closed).

(a) Let $d(x) = \|x - x^*\|^2$ for $x \in \mathbf{R}^n$ (d is the square of the distance to our given point x^*). Show that $\nabla d(x) = 2(x - x^*)$. (Hint: it may help to expand $d(x)$ in terms of coordinates of x.)

(b) Use the fact that C is convex to show that for any $c \in C$, $c - c^* \in A(C, c^*)$ (i.e., $c - c^*$ is an attainable direction at c^* for the set C). (Hint: use the line segment from c^* to c.)

(c) Considering the problem of minimizing $d(x)$ for $x \in C$, use (a) and (b) to show that $(c^* - x^*)^{\mathrm{T}}(c - c^*) \geq 0$ for all $c \in C$.

(d) Use (c) and the fact that C is a cone to show that $(c^* - x^*)^{\mathrm{T}}c^* = 0$. (Hint: use $\alpha < 1$ and $\alpha > 1$ with c^*.)

(e) Use (c) and (d) to show that $(c^* - x^*)^{\mathrm{T}}c \geq 0$ for all $c \in C$.

(f) Use (d) to show that $(c^* - x^*)^{\mathrm{T}}x^* < 0$. (Hint: $v^{\mathrm{T}}v > 0$ for any nonzero vector v.)

Thus, we have shown the existence of a vector a (specifically, $a = c^* - x^*$) such that $a^{\mathrm{T}}c \geq 0$ for all $c \in C$ (by (e)) but $a^{\mathrm{T}}x < 0$ (by (f)).