Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 3 Solutions

A3.1. Suppose we are working with the ordered basis
$A = (x^4 + x^3 + x^2 + x + 1, x^3 + x^2 + x + 1, x^2 + x + 1, x + 1, 1)$
of $P_4(\mathbf{R})$.

(a) If $[p(x)]_A = (5, -3, 0, 4, 0)$, what is $p(x)$?

(b) What is $[1 - 7x + 5x^2 - 2x^3 + 4x^4]_A$? (You can do this without using any matrices.)

Solution:] (a) $p(x) = 5(x^4 + x^3 + x^2 + x + 1) - 3(x^3 + x^2 + x + 1) + 4(x + 1) = 5x^4 + (5 - 3)x^3 + (5 - 3)x^2 + (5 - 3 + 4)x + (5 - 3 + 4) = 5x^4 + 2x^3 + 2x^2 + 6x + 6$.

(b) If the coordinates are $(a_1, a_2, a_3, a_4, a_5)$ then we have

$$\begin{aligned} a_1(x^4 + x^3 + x^2 + x + 1) + a_2(x^3 + x^2 + x + 1) + a_3(x^2 + x + 1) + a_4(x + 1) + a_5(1) \\ = 1 - 7x + 5x^2 - 2x^3 + 4x^4 \end{aligned}$$

from which, taking coefficients of powers of x, we get equations

$$\begin{array}{lccccccccccl} [x^4] & a_1 & & & & & & & & = & 4 & (1) \\ [x^3] & a_1 & + & a_2 & & & & & & = & -2 & (2) \\ [x^2] & a_1 & + & a_2 & + & a_3 & & & & = & 5 & (3) \\ [x^1] & a_1 & + & a_2 & + & a_3 & + & a_4 & & = & -7 & (4) \\ [x^0] & a_1 & + & a_2 & + & a_3 & + & a_4 & + a_5 & = & 1 & (5) \end{array}.$$

Subtracting the previous equation from each equation except the first, we get

$$\begin{array}{cccccccl} a_1 & & & & & = & 4 & (1) \\ & a_2 & & & & = & -6 & (2)-(1) \\ & & a_3 & & & = & 7 & (3)-(2) \\ & & & a_4 & & = & -12 & (4)-(3) \\ & & & & a_5 & = & 8 & (5)-(4) \end{array}$$

and so $[1 - 7x + 5x^2 - 2x^3 + 4x^4]_A = (4, -6, 7, -12, 8)$.

A3.2. Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ by
$T(a, b, c) = (a + b, c - 2a, 5c - b - a, b + 7c)$.

Solution:

$$[T] = [Te_1 \quad Te_2 \quad Te_3] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & -1 & 5 \\ 0 & 1 & 7 \end{bmatrix}.$$

A3.3. Find the matrix $[A]_C^B$ of the linear transformation $A : P_3(\mathbf{R}) \to \mathbf{R}^4$ by
$A(p(x)) = (p(0), p'(1), p''(2), p'''(3))$ (p', p'', p''' indicate first, second, third derivatives of p)
where $C = (1, 1 + x, x + x^2, x^2 + x^3)$ and B is the standard ordered basis in $\mathbf{R}^4$.

Solution: If we write $C = (c_1, c_2, c_3, c_4)$ then we have

$$\begin{aligned} [A]_C^B &= [[Ac_1]_B \quad [Ac_2]_B \quad [Ac_3]_B \quad [Ac_4]_B] = [(A(1) \quad A(1+x) \quad A(x+x^2) \quad A(x^2+x^3)] \\ &= \begin{bmatrix} 1|_{x=0} & 1+x|_{x=0} & x+x^2|_{x=0} & x^2+x^3|_{x=0} \\ 0|_{x=1} & 1|_{x=1} & 1+2x|_{x=1} & 2x+3x^2|_{x=1} \\ 0|_{x=2} & 0|_{x=2} & 2|_{x=2} & 2+6x|_{x=2} \\ 0|_{x=3} & 0|_{x=3} & 0|_{x=3} & 6|_{x=3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 14 \\ 0 & 0 & 0 & 6 \end{bmatrix}. \end{aligned}$$

A3.4. One application of alternative coordinate systems is that sometimes it is easy to compute the matrix of a linear transformation in a special coordinate system; then we can find the standard matrix using our conversion formulae.

Consider the linear operator R_π on $\mathbf{R}^3$ which reflects points in the plane π with equation $2x+3y+4z=0$.

(a) Find the matrix $[R_\pi]_C$ with respect to the ordered basis $C = ((2,3,4),(3,-2,0),(2,0,-1))$. (From the way we set up the equation of a plane, we know the first vector in C is normal to π. Show that the other two vectors in C lie in π. Then use these facts to find the matrix.)

(b) Use your answer to (a) to find the standard matrix of R_π.

(c) What is $R_\pi(7,1,2)$?

(d) Give a general formula for $R_\pi(x,y,z)$.

Solution: Let B be the standard ordered basis $B=(e_1,e_2,e_3)$ and write $C=(f_1,f_2,f_3)$.

(a) For f_2, $2x+3y+4z = 2(3)+3(-2)+4(0) = 6-6+0 = 0$, so $f_2 \in \pi$. For f_3, $2x+3y+4z = 2(2)+3(0)+4(-1) = 4+0-4 = 0$, so $f_3 \in \pi$.

Now since f_1 is normal to π, $R_\pi f_1 = -f_1$. Since f_2 and f_3 lie in π, $R_\pi f_2 = f_2$ and $R_\pi f_3 = f_3$. Therefore,

$$[R_\pi]_C = [R_\pi]_C^C = [[R_\pi f_1]_C \quad [R_\pi f_2]_C \quad [R_\pi f_3]_C] = [[-f_1]_C \quad [f_2]_C \quad [f_3]_C] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) We will use

$$[R_\pi] = [R_\pi]_B = [R_\pi]_B^B = [IR_\pi I]_B^B = [I]_C^B[R_\pi]_C^C[I]_B^C.$$

We have

$$P = [I]_C^B = [[If_1]_B \quad [If_2]_B \quad [If_3]_B] = [f_1 \quad f_2 \quad f_3] = \begin{bmatrix} 2 & 3 & 2 \\ 3 & -2 & 0 \\ 4 & 0 & -1 \end{bmatrix}$$

and therefore, using `LA`,

$$[I]_B^C = P^{-1} = \begin{bmatrix} 2/29 & 3/29 & 4/29 \\ 3/29 & -10/29 & 6/29 \\ 8/29 & 12/29 & -13/29 \end{bmatrix}.$$

So we compute

$$\begin{aligned}[R_\pi] = P[R_\pi]_C P^{-1} &= \begin{bmatrix} 2 & 3 & 2 \\ 3 & -2 & 0 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/29 & 3/29 & 4/29 \\ 3/29 & -10/29 & 6/29 \\ 8/29 & 12/29 & -13/29 \end{bmatrix} \\ &= \begin{bmatrix} 21/29 & -12/29 & -16/29 \\ -12/29 & 11/29 & -24/29 \\ -16/29 & -24/29 & -3/29 \end{bmatrix}.\end{aligned}$$

(c) We have that

$$R_\pi(7,1,2) = [R_\pi]\begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 21/29 & -12/29 & -16/29 \\ -12/29 & 11/29 & -24/29 \\ -16/29 & -24/29 & -3/29 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 103/29 \\ -121/29 \\ -142/29 \end{bmatrix}.$$

Hence, $R_\pi(7,1,2) = (\frac{103}{29}, -\frac{121}{29}, -\frac{142}{29})$.

(d) In general,

$$R_\pi(x,y,z) = [R_\pi]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21/29 & -12/29 & -16/29 \\ -12/29 & 11/29 & -24/29 \\ -16/29 & -24/29 & -3/29 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 21x-12y-16z \\ -12x+11y-24z \\ -16x-24y-3z \end{bmatrix}.$$

Hence, $R_\pi(x,y,z) = \frac{1}{29}(21x-12y-16z, -12x+11y-24z, -16x-24y-3z)$.

A3.5. Find a single 3×3 (real) matrix F so that left-multiplying a $3 \times n$ matrix A by F, forming FA, gives the same result as applying all four of the following elementary row operations, in the given order, to A.

(1) Add 2 times row 1 to row 3.
(2) Exchange rows 1 and 2.
(3) Add 4 times row 1 to row 3.
(4) Exchange rows 2 and 3.

Solution: Let ε_i represent the elementary row operation for step (i) above, then what we want is $FA = \varepsilon_4(\varepsilon_3(\varepsilon_2(\varepsilon_1(A))))$. But we know that for any $3 \times n$ matrix B, $\varepsilon_i(B) = E_i B$ where $E_i = \varepsilon_i(I_3)$ is the elementary matrix for ε_i. So we have

$$\varepsilon_4(\varepsilon_3(\varepsilon_2(\varepsilon_1(A)))) = E_4(E_3(E_2(E_1 A)))) = (E_4 E_3 E_2 E_1)A$$

So we want $F = E_4 E_3 E_2 E_1$. There are two ways to compute F.

First, we could just multiply the matrices:

$$\begin{aligned} F = E_4 E_3 E_2 E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Alternatively, $F = E_4 E_3 E_2 E_1 = E_4 E_3 E_2 E_1 I_3 = \varepsilon_4(\varepsilon_3(\varepsilon_2(\varepsilon_1(I_3))))$: just apply the elementary row operations in the given order to I_3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -\varepsilon_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} -\varepsilon_2 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} -\varepsilon_3 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 4 & 1 \end{bmatrix} -\varepsilon_4 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix} = F.$$