Math 175 Section 3 – Second-Year Accelerated Calculus – Spring 2003
Homework and Extra Material for Chapter 13

Homework problems

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
<th>Homework problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.1</td>
<td>821</td>
<td>5 7 13 15 23 29 33 35</td>
</tr>
<tr>
<td>13.2</td>
<td>828</td>
<td>3 5 7 11 13 19 25 31 33 41</td>
</tr>
<tr>
<td>13.3</td>
<td>836</td>
<td>1 3 7 11 17 21 25 31 35 41 45 49 51 55</td>
</tr>
<tr>
<td>13.4</td>
<td>843</td>
<td>1 5 9 11 15 19 23 27 31 35 39</td>
</tr>
<tr>
<td>13.5</td>
<td>852</td>
<td>3 9 19 25 29 31 51 61 X1 X2 X3 X4 X5</td>
</tr>
<tr>
<td>(X questions are given below on this handout)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.6</td>
<td>859</td>
<td>3 5 7 8 9 11 13 15 17 19 21–28 33 41 45</td>
</tr>
<tr>
<td>13.7</td>
<td>865</td>
<td>7 11 17 21 25 29 31 35 37 39 41 53 57 59</td>
</tr>
</tbody>
</table>

Answers for even-numbered problems from book


Extra questions for Section 13.5

13.5.X1. For each point $P$ and corresponding line $l$ below, determine whether $P$ is on $l$. If $P$ is not on $l$, find the closest point on $l$ to $P$, the distance from $l$ to $P$, and a second line through $P$ intersecting $l$ at right angles.

(a) $P = (1, 2, 3)$, $l: x = 2 + t$, $y = 2 - 3t$, $z = 5t$.
(b) $P = (1, 4, 2)$, $l: x = 3 - t$, $y = t + 2$, $z = 2t - 2$.
(c) $P = (1, 0, -1)$, $l: 5 - x = y/3 = (z - 1)/2$.

13.5.X2. For each pair of lines $l_1$ and $l_2$ below:
(i) Find the angle between them.
(ii) Determine if they are equal, parallel, intersecting, or skew.
(iii) If they are parallel (but not equal), find the distance between them, and a plane containing both of them. If they are intersecting (but not equal), find the point of intersection, a plane containing both of them, and a line intersecting both of them at right angles. If they are skew, find the pair of points, one on each line, that are as close as possible, the distance between the two lines, and a third line intersecting both lines at right angles.

(a) $l_1: \vec{r} = \langle 1, 2 - t, 3 + 2t \rangle$, $l_2: \vec{r} = \langle 5, 3 + 2t, 6 - 4t \rangle$.
(b) $l_1: x = 1 + t$, $y = 3t$, $z = 1 + 4t$, $l_2: x = 4t$, $y = 10 - t$, $z = 10 + 3t$.
(c) $l_1: (x - 1)/2 = (y - 2)/4 = (x - 3)/10$, $l_2: (x - 2)/3 = (y - 4)/6 = (z - 8)/15$.
(d) $l_1: x = t + 2$, $y = t + 11$, $z = t - 2$, $l_2: x = 5t - 8$, $y = 2t - 8$, $z = t + 2$.

13.5.X3. For each point $P$ and corresponding plane $\pi$ below, determine whether $P$ is on $\pi$, and then find a line through $P$ at right angles to $\pi$, the closest point on $\pi$ to $P$, and the distance from $\pi$ to $P$.

(a) $P = (2, 8, 5)$, $\pi: x - 2y - 2z = 1$.
(b) $P = (3, -2, 7)$, $\pi: -2x + 4y + 3z + 51 = 0$.
(c) $P = (1, 3, 7)$, $\pi: 2x + 5y + 3z = 38$. 


13.5.X4. For each line \( l \) and corresponding plane \( \pi \) below:
(i) Find the angle between \( l \) and \( \pi \).
(ii) Determine whether \( l \) is inside \( \pi \), intersects \( \pi \), or is parallel to and disjoint from \( \pi \).
(iii) If \( l \) intersects \( \pi \), find the point of intersection. If \( l \) is disjoint from \( \pi \), find the distance from \( \pi \) to \( l \).
(a) \( l: x = 5 - 2t, y = -1 - t, z = 16 + t, \pi: x + 2y + 4z = 25. \)
(b) \( l: (x - 1)/2 = (y - 2)/7 = -(z - 3)/4, \pi: 3x + 2y + 5z = 22. \)
(c) \( l: \vec{r} = (3t, 9 - t, 9 - 2t), \pi: -4x + 6y - 2z = 8. \)

13.5.X5. For each pair of planes \( \pi_1 \) and \( \pi_2 \) below:
(i) Find the angle between them.
(ii) Determine if they are equal, parallel, or intersecting.
(iii) If they are parallel (but not equal) find the distance between them. If they are intersecting, find an equation for the line of intersection.
(a) \( \pi_1: 2x - 4y + 8z = 6, \pi_2: -5x + 10y - 20z = -15. \)
(b) \( \pi_1: 2x - 4y + 8z = 10, \pi_2: -5x + 10y - 20z = -4. \)
(c) \( \pi_1: -5x + y - 3z = -13, \pi_2: 7x - 2y + 4z = 20. \)

Answers to extra questions for Section 13.5

Answer to 13.5.X1: (a) \( P \) is not on \( l \). Closest point is \( Q = (12/5, 4/5, 2) \) [hint: \( Q \) has \( \vec{P}Q \cdot \vec{v} = 0 \) where \( \vec{v} \) is parallel to \( l \)]. Distance is \( |\vec{P}Q| = \sqrt{110}/5 = \sqrt{22/5} \). One equation for the line is \( \vec{r} = \vec{OP} + t\vec{PQ} = (1, 2, 3) + t(7/5, -6/5, 1) \) (there are others).
(b) \( P \) is on \( l \).
(c) \( P \) is not on \( l \). Closest point is \( Q = (5, 0, 1) \). Distance is \( |\vec{P}Q| = \sqrt{20} = 2\sqrt{5} \). One equation for the line is \( \vec{r} = \vec{OP} + t\vec{PQ} = (1, 0, -1) + t(4, 0, 2) \) (there are others).

Note: Distances can be calculated without finding \( Q \), by using formula from 13.4.39, or also by using projections and Pythagoras’ Theorem.

Answer to 13.5.X2: (a) \( \vec{v}_1 = (0, -1, 2), \vec{v}_2 = (0, 2, -4) = -2\vec{v}_1 \) so parallel (but not equal from \( x \) values), angle is 0. \( \vec{n} = \vec{P_1P_2} \times \vec{v}_1 = (5, -8, -4) \). Distance \( |\vec{P_1P_2} \times \vec{v}_1|/|\vec{v}_1| = \sqrt{105}/\sqrt{5} = \sqrt{21} \). Plane \( 5x - 8y - 4z = -23 \).
(b) \( \vec{v}_1 = (1, 3, 4), \vec{v}_2 = (4, -1, 3) \), \( \cos \theta = 1/2, \theta = \pi/3 \), not parallel. Intersect at \( (4, 9, 13) \). \( \vec{n} = \vec{v}_1 \times \vec{v}_2 = (13, 13, -13) \) or use \( \vec{n}' = \vec{n}/13 = (1, 1, -1) \). Plane \( x + y - z = 0 \). One equation of third line is \( \vec{r} = (4, 9, 1) + t(1, 1, -1) \).
(c) \( \vec{v}_1 = (2, 4, 10), \vec{v}_2 = (3, 6, 15) = (3/2)\vec{v}_1 \) so parallel, angle is 0. Moreover \( \vec{P_1P_2} = (1, 2, 5) = (1/2)\vec{v}_1 \) so \( P_2 \) on \( l_1 \), lines are equal.
(d) \( \vec{v}_1 = (1, 1, 1), \vec{v}_2 = (5, 2, 1), \) not parallel, angle \( \theta = \cos^{-1}(8/(3\sqrt{10})) \). Do not intersect: \( x \) and \( y \) give \( t_1 = -25, t_2 = -3 \) then \( z \) disagrees. So lines skew. Closest points \( Q_1 = (-1, 8, -5) \) and \( Q_2 = (2, -4, 4) \) [hint: want \( Q_1Q_2 \cdot \vec{v}_1 = Q_1Q_2 \cdot \vec{v}_2 = 0 \). Distance \( |Q_1Q_2| = 3\sqrt{26} \). One equation of third line is \( \vec{r} = \vec{OQ_1} + t\vec{Q_1Q_2} = (1, 8, -5) + t(3, -12, 9) \).
Answer to 13.5.X3: (a) $P$ is not on $\pi$; one equation of line is $\vec{r} = \vec{OP} + t\vec{n}$ ($\vec{n}$ normal to $\pi$) = $\langle 2, 8, 5 \rangle + t(1, -2, -2)$. Hits $\pi$ when $t = 25/9$ at $Q = (43/9, 22/9, -5/9)$, closest point on $\pi$ to $P$; distance $|\overrightarrow{PQ}| = |t\vec{n}| = 25/3$.

(b) $P$ is not on $\pi$; one equation of line is $\vec{r} = \vec{OP} + t\vec{n} = \langle 3, -2, 7 \rangle + t(-2, 4, 3)$. Hits $\pi$ when $t = -2$ at $Q = (7, -10, 1)$, closest point on $\pi$ to $P$; distance $|\overrightarrow{PQ}| = |t\vec{n}| = 2\sqrt{29}$.

(c) $P$ is on $\pi$; one equation of line is $\vec{r} = \vec{OP} + t\vec{n} = \langle 1, 3, 7 \rangle + t(2, 5, 3)$. Since $P$ on $\pi$, closest point is $P$ itself, distance is 0.

Note: Distances could have been calculated without finding closest point, using formula from Example 8, p. 851.

Answer to 13.5.X4: (a) $\vec{v} = \langle -2, -1, 1 \rangle$, $\vec{n} = \langle 1, 2, 4 \rangle$, $\vec{v} \cdot \vec{n} = 0$, angle between line and normal is $\pi/2$, angle between line and plane is 0. $P_0 = (5, -1, 16)$ not in plane, so line parallel to but not in plane. Distance [hint: use formula from Example 8, p. 851] of $P_0$ and hence $l$ from $\pi$ is $|67 - 25|/\sqrt{21} = 2\sqrt{21}$.

(b) $\vec{v} = \langle 2, 7, -4 \rangle$, $\vec{n} = \langle 3, 2, 5 \rangle$, $\vec{v} \cdot \vec{n} = 0$, angle between line and normal is $\pi/2$, angle between line and plane is 0. $P_0 = (1, 2, 3)$ does lie in $\pi$, so $l$ is contained inside $\pi$.

(c) $\vec{v} = \langle 3, -1, -2 \rangle$, $\vec{n} = \langle -4, 6, -2 \rangle$, angle between $\vec{v}$ and $\vec{n}$ is $\cos^{-1}(-1/2) = 2\pi/3$. So angle between line and plane is $\pi/6$, line must meet plane. Intersection point is $(6, 7, 5)$ when $t = 2$.

Answer to 13.5.X5: (a) $\vec{n}_1 = \langle 2, -4, 8 \rangle$, $\vec{n}_2 = \langle -5, 10, -20 \rangle = \langle -5/2 \rangle \vec{n}_1$ so the two planes are parallel, angle is 0. For every point on $\pi_1$, $-5x + 10y - 20z = (-5/2)(2x - 4y + 8z) = -5/2(6) = -15$, so $\pi_1$ is contained in $\pi_2$, so the planes are equal.

(b) $\vec{n}_1 = \langle 2, -4, 8 \rangle$, $\vec{n}_2 = \langle -5, 10, -20 \rangle = \langle -5/2 \rangle \vec{n}_1$ so the two planes are parallel, angle is 0. For every point on $\pi_1$, $-5x + 10y - 20z = (-5/2)(2x - 4y + 8z) = (-5/2)(10) = -25 \neq -4$, so $\pi_1$ is disjoint from $\pi_2$: they are not equal. Taking the point $P_2 = (0, -2/5, 0)$ on $\pi_2$ and using formula from Example 8, p. 851, distance of $P_2$ from $\pi_1$ and hence $\pi_2$ from $\pi_1$ is $|8/5 - 10|/\sqrt{84} = \sqrt{21}/5$.

(c) $\vec{n}_1 = \langle -5, 1, -3 \rangle$, $\vec{n}_2 = \langle 7, -2, 4 \rangle$, angle between normals is $\cos^{-1}(-49/(\sqrt{35\sqrt{69}}))$, so ignoring sign, angle between planes is $\cos^{-1}(49/\sqrt{2415})$. Planes must intersect. Line has $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -1, 3 \rangle$. For point on line put $x = 0$, gives $y = -4$, $z = 3$, so one equation of line is $\vec{r} = \langle 0, -4, 3 \rangle + t\langle -2, -1, 3 \rangle$.

Summary: What you should be able to do with points, lines and planes

Given a point $P$ and a line $l$:
- Determine if $P$ is on $l$
- Find distance of $P$ from $l$
- Find closest point on $l$ to $P$
- If $P$ not on $l$:
  - Find line through $P$ intersecting $l$ perpendicularly

Given two lines $l_1$ and $l_2$:
- Find angle between $l_1$ and $l_2$
- Determine if $l_1$ and $l_2$ equal, parallel, intersecting, or skew
- If parallel (but not equal):
  - Find distance between $l_1$ and $l_2$
  - Find plane containing $l_1$ and $l_2$
- If intersecting (but not equal):
  - Find point of intersection
Find plane containing $l_1$ and $l_2$
Find line intersecting both $l_1$ and $l_2$ at right angles

If skew:
  Find distance between $l_1$ and $l_2$
  Find closest points
  Find line intersecting both $l_1$ and $l_2$ at right angles

Given a point $P$ and a plane $\pi$:
  Determine if $P$ is on $\pi$
  Find line through $P$ perpendicular to $\pi$
  Find distance of $P$ from $\pi$
  Find closest point on $\pi$ to $P$

Given a line $l$ and a plane $\pi$:
  Find angle between $l$ and $\pi$
  Determine if $l$ is inside, intersects, or is disjoint from $\pi$
  If intersects:
    Find point of intersection
  If disjoint:
    Find distance between $l$ and $\pi$

Given two planes $\pi_1$ and $\pi_2$:
  Find angle between $\pi_1$ and $\pi_2$
  Determine if $\pi_1$ and $\pi_2$ equal, parallel or intersecting
  If parallel (but not equal):
    Find distance between $\pi_1$ and $\pi_2$
  If intersecting (but not equal):
    Find equation of line of intersection