Math 150b Section 9 – First-Year Calculus II – Spring 2004
Solutions to Assignment 8

1. Evaluate \( \int \frac{3x^5 - 3x^4 - 17x^3 + 39x^2 - 77x + 81}{(x-2)(x^2 + x - 6)} \, dx \).

\[ \text{Solution:} \] Temporarily, we expand the denominator (to make long division easier), so the integrand is
\[ \frac{3x^5 - 3x^4 - 17x^3 + 39x^2 - 77x + 81}{x^3 - x^2 - 8x + 12}. \]
We next perform long division:
\[
\begin{array}{cccc}
& 3x^2 & +7 \\
\hline
x^3 & -x^2 & -8x & +12 \\
3x^5 & -3x^4 & -17x^3 & +39x^2 & -77x & +81 \\
3x^5 & -3x^4 & -24x^3 & +36x^2 & & \\
7x^3 & +3x^2 & -77x & +81 & \\
7x^3 & -7x^2 & -56x & +84 & \\
10x^2 & -21x & -3 & \\
\end{array}
\]
(remember that we subtract the new line from the previous line at each step). Thus the integrand is
\[
3x^2 + 7 + \frac{10x^2 - 21x - 3}{x^3 - x^2 - 8x + 12} = 3x^2 + 7 + \frac{10x^2 - 21x - 3}{(x-2)(x^2 + x - 6)}
\]
\[
= 3x^2 + 7 + \frac{10x^2 - 21x - 3}{(x-2)(x^2 + x - 6)}
\]
\[
= 3x^2 + 7 + \frac{10x^2 - 21x - 3}{(x-2)(x+3)}.
\]
We wish to express the last term as a sum of partial fractions. Notice that the denominator is a product of linear factors, where \((x-2)\) is repeated. So we write
\[
\frac{10x^2 - 21x - 3}{(x-2)^2(x+3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3}
\]
and we must solve for \(A\), \(B\) and \(C\).
\[
\begin{align*}
\frac{10x^2 - 21x - 3}{(x-2)^2(x+3)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3} \\
10x^2 - 21x - 3 &= A(x-2)(x+3) + B(x+3) + C(x-2)^2 \\
10x^2 - 21x - 3 &= A(x-2)(x+3) + B(x+3) + C(x-2)^2 \\
10x^2 - 21x - 3 &= A(x-2)(x+3) + B(x+3) + C(x-2)^2
\end{align*}
\]
so the numerators must be equal, i.e.,
\[
10x^2 - 21x - 3 = A(x-2)(x+3) + B(x+3) + C(x-2)^2 \tag{1}
\]
Equation (1) must hold for all values of \(x\). In particular, it holds for \(x = 2\). Substituting, we have
\[
10(2)^2 - 21(2) - 3 = A \cdot 0 \cdot 5 + B \cdot 5 + C \cdot 0^2
\]
\[
-5 = B \cdot 5
\]
\[
-1 = B
\]
1
so $B = -1$. Also, since (1) holds for all values of $x$, it holds for $x = -3$. Substituting, we have

\[
10(-3)^2 - 21(-3) - 3 = A \cdot (-5) \cdot 0 + B \cdot 0 + C \cdot (-5)^2
\]

\[
150 = C \cdot 25
\]

\[
6 = C
\]

so $C = 6$.

Now we substitute $B = -1$ and $C = 6$ into (1) to obtain a new equation

\[
10x^2 - 21x - 3 = A(x - 2)(x + 3) + (-1)(x + 3) + 6(x - 2)^2
\]

This equation holds for all values of $x$ as well, so it holds for $x = 0$. We substitute to get

\[
10(0)^2 - 21(0) - 3 = A \cdot (-2) \cdot 3 - 3 + 6(-2)^2
\]

\[
-3 = -6A + 21
\]

\[
-24 = -6A
\]

\[
A = 4
\]

so $A = 4$.

Since $A = 4$, $B = -1$ and $C = 6$, we have

\[
\frac{10x^2 - 21x - 3}{(x - 2)^2(x + 3)} = \frac{4}{x - 2} - \frac{1}{(x - 2)^2} + \frac{6}{x + 3}
\]

Therefore

\[
\int \frac{3x^5 - 3x^4 - 17x^3 + 39x^2 - 77x + 81}{(x - 2)(x^2 + x - 6)} \, dx = \int \left( 3x^2 + 7 + \frac{4}{x - 2} - \frac{1}{(x - 2)^2} + \frac{6}{x + 3} \right) \, dx
\]

\[
= x^3 + 7x + 4\ln|x - 2| + \frac{1}{x - 2} + 6\ln|x + 3| + C.
\]

2. Evaluate \( \int \frac{dx}{x^{1/7} + x^{2/7} + 9x^{2/7} + 9x^{1/7}} \). [Hint: substitute $u = x^{1/7}$.]

Solution: Let $u = x^{1/7}$, so that $du = \frac{1}{7}x^{-6/7} \, dx$. Then we have

\[
\int \frac{dx}{x^{1/7} + x^{2/7} + 9x^{2/7} + 9x^{1/7}} = \int \frac{7u^{6/7}(\frac{1}{7}x^{-6/7}) \, dx}{x^{4/7} + x^{3/7} + 9x^{2/7} + 9x^{1/7}}
\]

\[
= \int \frac{7u^6 \, du}{u^4 + u^3 + 9u^2 + 9u}
\]

\[
= \int \frac{7u^5 \, du}{u^3 + u^2 + 9u + 9}
\]

\[
= 7 \int \frac{u^5 \, du}{u^3 + u^2 + 9u + 9}.
\]

We next perform long division

\[
\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr
\end{array}
\]
Therefore the integrand is
\[
\frac{u^2 - u - 8 + 8u^2 + 81u + 72}{u^3 + u^2 + 9u + 9} = u^2 - u - 8 + \frac{8u^2 + 81u + 72}{u^2(u + 1) + 9(u + 1)}
\]
\[
= u^2 - u - 8 + \frac{8u^2 + 81u + 72}{(u + 1)(u^2 + 9)}.
\]

We wish to express the last term as a sum of partial fractions. Notice that the denominator is the product of a linear factor and an irreducible quadratic factor. So we write
\[
\frac{8u^2 + 81u + 72}{(u + 1)(u^2 + 9)} = \frac{A}{u + 1} + \frac{Bu + C}{u^2 + 9}
\]
and we must solve for \(A\), \(B\) and \(C\).

\[
8u^2 + 81u + 72 = A(u^2 + 9) + (Bu + C)(u + 1)
\]
so the numerators must be equal, i.e.,
\[
8u^2 + 81u + 72 = A(u^2 + 9) + (Bu + C)(u + 1) \quad (2)
\]
Equation (2) holds for every value of \(u\). In particular, it holds for \(u = -1\). Substituting, we have
\[
8(-1)^2 + 81(-1) + 72 = A \cdot 10 + (B(-1) + C) \cdot 0
\]
\[-1 = A \cdot 10
\]
\[-\frac{1}{10} = A
\]
so \(A = -\frac{1}{10}\). We substitute \(A = -\frac{1}{10}\) into (2) to get a new equation
\[
8u^2 + 81u + 72 = -\frac{1}{10}(u^2 + 9) + (Bu + C)(u + 1)
\]
\[
8u^2 + 81u + 72 = -\frac{1}{10}u^2 - \frac{9}{10} + Bu^2 + Bu + Cu + C
\]
\[
\frac{81}{10}u^2 + 81u + \frac{729}{10} = Bu^2 + (B + C)u + C.
\]
Now the coefficients of like powers must be equal, so, in particular, the coefficient of \(u^2\) is \([u^2] = \frac{81}{10} = B\), which gives \(B = \frac{81}{10}\). Also, the constant terms must be equal, so \(\frac{729}{10} = C\). (As a check, we should have \([u] = 81 = B + C\), and, in fact, \(B + C = \frac{81}{10} + \frac{729}{10} = 81\).)

Thus we have \(A = -\frac{1}{10}\), \(B = \frac{81}{10}\) and \(C = \frac{729}{10}\), so
\[
\frac{8u^2 + 81u + 72}{(u + 1)(u^2 + 9)} = -\frac{1}{10} + \frac{81}{10}u + \frac{729}{10}u^2 + 9
\]
\[
= -\frac{1}{10} \left( \frac{1}{u + 1} \right) + \frac{81}{10} \left( \frac{u + 9}{u^2 + 9} \right).
\]
Therefore our original integral is equal to

\[
7 \int \frac{u^5 \, du}{u^3 + u^2 + 9u + 9}
= 7 \int u^2 - u - 8 - \frac{1}{10} \left( \frac{1}{u + 1} \right) + \frac{81}{10} \left( \frac{u + 9}{u^2 + 9} \right) \, du
= 7 \int u^2 - u - 8 - \frac{1}{10} \left( \frac{1}{u + 1} \right) + \frac{81}{10} \left( \frac{u}{u^2 + 9} \right) + \frac{729}{10} \left( \frac{1}{u^2 + 9} \right) \, du
= 7 \left[ \frac{1}{3} u^3 - \frac{1}{2} u^2 - 8u - \frac{1}{10} \ln |u + 1| + \frac{81}{10} \int \frac{u}{u^2 + 9} \, du + \frac{729}{10} \left( \frac{1}{3} \tan^{-1} \left( \frac{u}{3} \right) \right) \right]
= 7 \left[ \frac{1}{3} u^3 - \frac{1}{2} u^2 - 8u - \frac{1}{10} \ln |u + 1| + \frac{81}{20} \int \frac{1}{w} \, dw + \frac{243}{10} \tan^{-1} \left( \frac{u}{3} \right) \right] + C
= 7 \left[ \frac{1}{3} u^3 - \frac{1}{2} u^2 - 8u - \frac{1}{10} \ln |u + 1| + \frac{81}{20} \ln(u^2 + 9) + \frac{243}{10} \tan^{-1} \left( \frac{u}{3} \right) \right] + C
\]

(At (3), we let \( w = u^2 + 9 \), with \( dw = 2u \, du \).)

Thus

\[
\int \frac{dx}{x^{4/7} + x^{3/7} + 9x^{2/7} + 9x^{1/7}}
= 7 \left[ \frac{1}{3} x^{3/7} - \frac{1}{2} x^{2/7} - 8x^{1/7} - \frac{1}{10} \ln |x^{1/7} + 1| + \frac{81}{20} \ln(x^{2/7} + 9) + \frac{243}{10} \tan^{-1} \left( \frac{x^{1/7}}{3} \right) \right] + C.
\]