1. Use logarithmic differentiation to find the derivative of
\[ y = \frac{(x^3 + 7x^2 + 2x + 4)^{37} \cos^{12} x}{(\ln x)^{23}}. \]

**Solution:** We take logarithms of the absolute values of both sides of the equation:
\[ \ln |y| = \ln \left| \frac{(x^3 + 7x^2 + 2x + 4)^{37} \cos^{12} x}{(\ln x)^{23}} \right|. \]

Using properties of logarithms, we have that
\[ \ln |y| = 37 \ln |x^3 + 7x^2 + 2x + 4| + 12 \ln |\cos x| - 23 \ln |\ln x|. \]

Differentiating implicitly with respect to \( x \) gives
\[ \frac{1}{y} \frac{dy}{dx} = 37 \cdot \frac{3x^2 + 14x + 2}{x^3 + 7x^2 + 2x + 4} + 12 \cdot \frac{-\sin x}{\cos x} - 23 \cdot \frac{1}{\ln x} \quad \text{(by the Chain Rule)}. \]

Solving for \( \frac{dy}{dx} \), we get
\[ \frac{dy}{dx} = y \left( 37 \cdot \frac{3x^2 + 14x + 2}{x^3 + 7x^2 + 2x + 4} - 12 \tan x - \frac{23}{x \ln x} \right). \]

We are given an expression for \( y \), so we substitute to get
\[ \frac{dy}{dx} = \left( \frac{(x^3 + 7x^2 + 2x + 4)^{37} \cos^{12} x}{(\ln x)^{23}} \right) \left( 37 \cdot \frac{3x^2 + 14x + 2}{x^3 + 7x^2 + 2x + 4} - 12 \tan x - \frac{23}{x \ln x} \right). \]

2. Suppose we have a coordinate system where the \( x \)-axis and \( y \)-axis are marked with scales in metres (m). A tank is in the shape of the solid obtained by rotating the area bounded by the \( y \)-axis, the curve \( y = 4 - \sqrt{4 - x^2} - 9 \) (where \( x > 0 \)), and the lines \( y = 0 \) and \( y = 4 \) around the \( y \)-axis. It is filled with a light oil weighing 6000 N/m\(^3\). The liquid is pumped out through an outlet at the top of the tank. Find the work done.

**Solution:** We use the formula developed in class for situations like this – the work done is given by
\[ W = \int_{a}^{b} \rho d(y) A(y) \, dy \]

where \( \rho \) is the density in N/m\(^3\), \( d(y) \) is the distance by which the liquid at position \( y \) has to be lifted, and \( A(y) \) is the cross-sectional area at \( y \). We integrate with respect to \( y \) because we want to measure depths of the oil in the tank.

Here \( \rho = 6000 \) N/m\(^3\) since this is the density of the oil. From the graphs, we see that the lines \( y = 0 \) and \( y = 4 \) force us to integrate \( y \) from 0 to 4. For a fixed \( y \), the distance that the layer of oil at height \( y \) has to be raised is \( 4 - y \) m, since the top of the tank is at height 4 m. Therefore \( d(y) = 4 - y \).
In order to find $A(y)$, we need to find the radius of the cross-sectional circle at height $y$. We solve the equation

$$y = 4 - \sqrt{\frac{4}{x^2} - 9}$$

for $x$.

$$y = 4 - \sqrt{\frac{4}{x^2} - 9}$$

$$y - 4 = -\sqrt{\frac{4}{x^2} - 9}$$

$$(4 - y)^2 = \frac{4}{x^2} - 9$$

$$(4 - y)^2 + 9 = \frac{4}{x^2}$$

$$x^2 = \frac{4}{(4 - y)^2 + 9}$$

$$x = \sqrt{\frac{4}{(4 - y)^2 + 9}}$$

(we are considering $x > 0$).

Therefore the radius is

$$\sqrt{\frac{4}{(4 - y)^2 + 9}}$$

so

$$A(y) = \pi \left(\sqrt{\frac{4}{(4 - y)^2 + 9}}\right)^2 = \frac{4\pi}{(4 - y)^2 + 9}.$$ 

Thus the work done is

$$W = \int_0^4 \rho d(y)A(y) \, dy = \int_0^4 6000 \cdot (4 - y) \cdot \frac{4\pi}{(4 - y)^2 + 9} \, dy$$

$$= 24000\pi \int_0^4 \frac{4 - y}{(4 - y)^2 + 9} \, dy$$

$$= -12000\pi \int_{y=4}^{y=0} \frac{1}{u + 9} \, du$$

(letting $u = (4 - y)^2$ so $du = -2(4 - y) \, dy$)

$$= -12000\pi \int_{u=16}^{u=0} \frac{1}{u + 9} \, du$$

(when $y = 0, u = 16$, and when $y = 4, u = 0$)

$$= -12000\pi \ln(u + 9) \bigg|_{16}^{0}$$

$$= -12000\pi \ln(9) - (-12000\pi \ln(25))$$

$$= 12000\pi (\ln(25) - \ln(9))$$

$$= 12000\pi \ln\left(\frac{25}{9}\right)$$

$$= 12000\pi \ln\left(\frac{5}{3}\right)^2$$

$$= 24000\pi \ln\left(\frac{5}{3}\right) \text{ N-m.}$$
3. The area bounded by \( y = \sqrt{\ln x}, \ y = 1 \) and \( x = e^4 \) is rotated around the \( x \)-axis. Find the volume of the resulting solid by using cylindrical shells.

**Solution:** Since we are rotating around the \( x \)-axis and using cylindrical shells, we will need to integrate with respect to \( y \). From the graphs of the three curves, we see that we need to find the point where \( y = \sqrt{\ln x} \) and \( x = e^4 \) intersect. The \( x \)-coordinate of this intersection point is \( e^4 \), and the \( y \)-coordinate is \( \sqrt{\ln e^4} = \sqrt{4} = 2 \) (since the \( y \)-coordinate is positive), so these curves intersect at \((e^4, 2)\). Thus we will need to integrate from \( y = 1 \) to \( y = 2 \).

We also need to solve the equation \( y = \sqrt{\ln x} \) for \( x \), since our height function will be in terms of \( y \). We have
\[
\begin{align*}
y &= \sqrt{\ln x} \\
y^2 &= \ln x \\
e^{y^2} &= e^{\ln x} \\
e^{y^2} &= x
\end{align*}
\]
so \( x = e^{y^2} \).

Now \( r(y) = y \) (since we’re rotating around the \( x \)-axis), and \( h(y) = e^4 - e^{y^2} \) (since \( e^4 \) is to the right of \( e^{y^2} \)). Thus the volume of the solid is
\[
\int_1^2 2\pi r(y)h(y) \, dy = \int_1^2 2\pi y(e^4 - e^{y^2}) \, dy
\]
\[
= 2\pi e^4 \int_1^2 y \, dy - 2\pi \int_1^2 ye^{y^2} \, dy
\]
\[
= 2\pi e^4 \left[ \frac{1}{2} y^2 \right]_1^2 - 2\pi \int_{y=1}^{y=2} e^{\frac{1}{2}u} \, du \quad \text{(letting } u = y^2 \text{ so } du = 2ydy)\]
\[
= \pi e^4 y^2 \bigg|_1^2 - \pi e^{u^2} \bigg|_1^2
\]
\[
= \pi e^4 y^2 \bigg|_1^2 - \pi e^{u^2} \bigg|_1^2
\]
\[
= \left( \pi e^4 (2)^2 - \pi e^4 (1)^2 \right) - \left( \pi e^{(2)^2} - \pi e^{(1)^2} \right)
\]
\[
= \left( 4\pi e^4 - \pi e^4 \right) - \left( \pi e^4 - \pi e \right)
\]
\[
= 2\pi e^4 + \pi e
\]
\[
= \pi e(2e^3 + 1).
\]