UNBOUNDED ASYMMETRY OF STRETCH FACTORS

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Abstract. A result of Handel–Mosher guarantees that the ratio of logarithms of stretch factors of any fully irreducible automorphism of the free group F_N and its inverse is bounded by a constant C_N. In this short note we show that this constant C_N cannot be chosen independent of N.

Let F_N be the free group of rank $N \geq 2$. An outer automorphism $\varphi \in \mathrm{Out}(F_N)$ is said to be *fully irreducible* if no power of φ preserves the conjugacy class of any proper free factor of F_N. In this case φ has a well defined *stretch factor* $\lambda(\varphi)$, which, for any non-φ–periodic conjugacy class α in F_N and a free basis X of F_N, is given by

$$\lambda(\varphi) = \lim_{n\to\infty} \sqrt[n]{\|\varphi^n(\alpha)\|_X},$$

where $\|\cdot\|_X$ denotes cyclically reduced word length with respect to X. As was observed in [BH] (see also [HM2]), there exist fully irreducible elements $\varphi \in \mathrm{Out}(F_N)$ with the property that φ and φ^{-1} have *different* stretch factors:

$$\lambda(\varphi) \neq \lambda(\varphi^{-1}).$$

However, the following result from [HM1] describes the extent to which they can differ. To state their result precisely, let $N \geq 2$ and set

$$C_N = \sup_{\varphi} \frac{\log(\lambda(\varphi))}{\log(\lambda(\varphi^{-1}))},$$

where φ ranges over all fully irreducible elements of $\mathrm{Out}(F_N)$.

Theorem 1 (Handel-Mosher)**.** *For $N \geq 2$, $C_N < \infty$.*

An alternate proof of this result was more recently given by Algom-Kfir and Bestvina [AKB]. While the proofs of this theorem appeal to the fact that N is fixed, it is not clear that this dependence is necessary. In this short note, we prove that in fact it is.

Theorem 2. *With $\{C_N\}_{N\geq 2}$ defined as above, $\limsup_{N\to\infty} C_N = \infty$.*

Proof. The proof will appeal to a construction and analysis carried out in [DKL1] and [DKL2]. To that end, let $F_3 = \langle a, b, c\rangle$ and consider the element $\varphi \in \mathrm{Aut}(F_3)$ defined by

$$\varphi(a) = b\,, \quad \varphi(b) = b^{-1}a^{-1}bac\,, \quad \varphi(c) = a.$$

It was shown in [DKL1, Example 5.5] that φ is fully irreducible. Next, let

$$G = F_3 \rtimes_\varphi \mathbb{Z} = \langle a, b, c, r \mid r^{-1}\,x\,r = \varphi(x) \text{ for all } g \in F_3\rangle$$

be the free-by-cyclic group determined by φ, and let $u_0\colon G \to \mathbb{Z}$ in $\mathrm{Hom}(G;\mathbb{R}) = H^1(G;\mathbb{R})$ be the associated homomorphism obtained by sending r to 1 and all other generators to 0.

In [DKL1], we construct a cone $\mathcal{A} \subset H^1(G;\mathbb{R})$ containing u_0 with the property that every other primitive integral element $u \in \mathcal{A}$ has kernel $\ker(u)$ a finitely generated free group. The action of $u(G) = \mathbb{Z}$ on $\ker(u)$ is generated by a *monodromy automorphism* $\varphi_u \in \mathrm{Aut}(\ker(u))$ determining an expression of G as a semidirect

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product $G \cong \ker(u) \rtimes_{\varphi_u} \mathbb{Z}$ with associated homomorphism u. One of the main results of [DKL1] is that all such φ_u are fully irreducible.

In [DKL2], we construct a strictly larger open, convex cone $\mathcal{A} \subsetneq \mathcal{S} \subset H^1(G;\mathbb{R})$ and a function

$$\mathfrak{H}\colon \mathcal{S} \to \mathbb{R}$$

that is convex, real analytic, and homogeneous of degree -1 (i.e., $\mathfrak{H}(tu) = \frac{1}{t}\mathfrak{H}(u)$) such that

$$\log(\lambda(\varphi_u)) = \mathfrak{H}(u)$$

for any primitive integral class $u \in \mathcal{A}$. In fact this holds for all primitive integral $u \in \mathcal{S}$ with the appropriate interpretation of $\lambda(\varphi_u)$. We also show that $\mathcal{S}$ is the cone on the component of the BNS-invariant $\Sigma(G)$ [BNS] containing u_0 [DKL2, Theorem I] and that $\mathcal{A}$ lies over the symmetrized BNS-invariant (that is, both $\mathcal{A}$ and $-\mathcal{A}$ project into $\Sigma(G)$) [DKL2, Corollary 13.7]. In fact, a key result of Bieri–Neumann-Strebel is that an integral class $u \in \mathrm{Hom}(G;\mathbb{Z})$ has $\ker(u)$ finitely generated if and only if both u and $-u$ lie in the $\Sigma(G)$ [BNS].

The the homomorphism $-u_0$ has $\ker(-u_0) = \ker(u_0) = F_N$ and associated monodromy φ^{-1}, thus expressing G as $F_N \rtimes_{\varphi^{-1}} \mathbb{Z}$. Since φ^{-1} is also fully irreducible, the main result of [DKL2] provides another open, convex cone $\mathcal{S}_- \subset H^1(G;\mathbb{R})$ containing $-u_0$ and a corresponding convex, real analytic, homogeneous of degree -1 function $\mathfrak{H}_-\colon \mathcal{S}_- \to \mathbb{R}$. Since $-\mathcal{A}$ projects into $\Sigma(G)$ and $\mathcal{S}_-$ is the cone on the component of $\Sigma(G)$ containing $-u_0$, we see that $-\mathcal{A} \subset \mathcal{S}_-$. Thus $\mathfrak{H}_-$ calculates the inverse stretch factors

$$\mathfrak{H}_-(-u) = \log(\lambda(\varphi_u^{-1}))$$

for all primitive integral $u \in \mathcal{A}$.

Example 8.3 of [DKL2] exhibits a primitive integral class $u_1 \in \mathcal{S}$ which lies on the boundary of $\mathcal{A}$ (see [DKL2, Figure 8]) for which $\ker(u_1)$ is *not* finitely generated. It follows that $-u_1$ is *not* in the BNS–invariant. The key observation is that $-u_1$ then necessarily lies on the boundary of $\mathcal{S}_-$ (since $-u_1 \in \overline{-\mathcal{A}} \subset \overline{\mathcal{S}_-}$ but $-u_1 \notin \mathcal{S}_-$). Let $\{u_n\}_{n=2}^{\infty} \subset \mathcal{A}$ be primitive integral classes protectively converging to u_1. That is, there exists $\{t_n\}_{n=2}^{\infty} \subset \mathbb{R}$ so that $\lim_{n\to\infty} t_n u_n = u_1$. Since this convergence occurs inside $\mathcal{S}$, it follows that

$$\lim_{n\to\infty} \mathfrak{H}(t_n u_n) = \mathfrak{H}(u_1) < \infty.$$

On the other hand, since $\lim_{n\to\infty} -t_n u_n = -u_1 \in \partial\mathcal{S}_-$, it follows from [DKL2, Theorem F] that

$$\lim_{n\to\infty} \mathfrak{H}_-(-t_n u_n) = \infty.$$

Therefore, appealing to the homogeneity of $\mathfrak{H}$ and $\mathfrak{H}_-$, we have

$$\lim_{n\to\infty} \frac{\log(\lambda(\varphi_{u_n}^{-1}))}{\log(\lambda(\varphi_{u_n}))} = \lim_{n\to\infty} \frac{\mathfrak{H}_-(-u_n)}{\mathfrak{H}(u_n)} = \lim_{n\to\infty} \frac{\mathfrak{H}_-(-t_n u_n)}{\mathfrak{H}(t_n u_n)} = \infty. \qquad \square$$

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