Math 3890: Dynamical Systems – Assignment 10

Due in-class on Wednesday, April 10

This assignment has 6 questions for a total of 60 points.

1. (10 points) Let $\sigma\colon \Sigma_A \to \Sigma_A$ be the topological Markov chain associated a transition matrix A. Let $P_m(\sigma)$ denote the number of points $\omega \in \Sigma_A$ fixed by σ^m. If A is aperiodic, prove that the entropy of σ_A satisfies
$$h(\sigma_A) = \lim_{m\to\infty} \frac{\log P_m(\sigma_A)}{m}.$$
[Note: Aperiodicity is not actually necessary, but may allow for a simpler proof.]

2. (10 points) Find the smallest *positive* value of $h(\sigma_A)$ for a topological Markov chain with two states (that is, a 2×2 transition matrix A).

3. (10 points) Let $f\colon X \to X$ be a continuous map of a compact metric space X. Prove that
$$h(f^m) = mh(f) \quad \text{for all } m \in \mathbb{N}.$$

4. (10 points) Suppose $f\colon X \to X$ and $g\colon Y \to Y$ are continuous maps of a compact metric spaces X and Y. Consider the product space $X \times Y$, say with the metric
$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$
Calculate the entropy of the map $f \times g\colon X \times Y \to X \times Y$, defined as $(f \times g)(x, y) = (f(x), g(y))$, in terms of $h(f)$ and $h(g)$.

5. (10 points) Construct a map with positive topological entropy that has no periodic points.

6. (10 points) Let $X = \{z \in \mathbb{C} : |z| \le 1\}$ be the closed unit disk. For $0 \le \lambda \le 1$ define a map $f_\lambda\colon X \to X$ in polar coordinates $z = re^{i\theta}$ (with $r, \theta \in \mathbb{R}$) by $f_\lambda(re^{i\theta}) = \lambda re^{2i\theta}$. Show that $h(f_1) = \log 2$ and that $h(f_\lambda) = 0$ for $0 \le \lambda < 1$.