Math 3890: Dynamical Systems – Assignment 9

Due in-class on Wednesday, April 3

This assignment has 5 questions for a total of 55 points.

If $g\colon X \to X$ and $f\colon Y \to Y$ are self-maps of metric spaces, we say that f *is a factor of* g if there is a continuous surjection $h\colon X \to Y$ such that $h \circ g = f \circ h$ (that is, if f is semi-conjugate to g).

1. (10 points) For $k \in \mathbb{N}$, let $\sigma_k\colon \Sigma_k \to \Sigma_k$ denote the left shift on the full shift space $\Sigma_k = \{1, \ldots, k\}^{\mathbb{N}}$. Prove that for $m < n$, the shift σ_m is a factor of the shift σ_n.

2. (10 points) Show that the topological Markov chains $\sigma_A\colon \Sigma_A \to \Sigma_A$ and $\sigma_B\colon \Sigma_B \to \Sigma_B$ associated to the matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are topologically conjugate.

Let $\sigma\colon \Sigma_A \to \Sigma_A$ be the topological Markov chain associated a transition matrix A. Let $P_m(\sigma)$ denote the number of points $\omega \in \Sigma_A$ fixed by σ^m. The zeta function of σ is the "generating function" of $P_m(\sigma)$. That is, we formally define the **zeta function** of σ to be the power series

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} P_m(\sigma)\right), \qquad \text{where } z \in \mathbb{C}.$$

3. (10 points) Show that the zeta function can be rewritten as infinite product

$$\zeta(z) = \prod_{\gamma} \left(1 - z^{|\gamma|}\right)^{-1}$$

where $\gamma = \{\omega, \sigma(\omega), \ldots, \sigma^{m-1}(\omega)\}$ denotes a periodic orbit of *prime* period $|\gamma| := m$. (That is, the infinite product over all finite orbits of σ). [*Hint:* Use $\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) = \exp\left(-\log(1-z)\right)$.]

4. (10 points) Show that the zeta function can be written as $\zeta(z) = 1/\det(I - zA)$. Deduce that $\zeta(z)$ is a rational function.

5. Calculate the zeta function $\zeta(z)$ for the following subshifts $\sigma\colon \Sigma_A \to \Sigma_A$.
   (a) (5 points) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, i.e., the "full shift on two symbols."
   (b) (5 points) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
   (c) (5 points) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.