Math 3890: Dynamical Systems – Assignment 8

Due in-class on Wednesday, March 27

This assignment has 5 questions for a total of 60 points.

Recall that Sarkovskii's theorem says that if a continuous self-map $f\colon I \to I$ of an interval $I \subset \mathbb{R}$ has a point with period n, then it has a periodic point for each period that follows n in the ordering of the natural numbers given by:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots$$
$$\cdots \rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd 2^3\cdot 7 \rhd \cdots \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2^2 \rhd 2 \rhd 1.$$

1. (10 points) In lecture we proved Sarkovskii's Theorem in the cases that n is an odd or a power of 2. Prove the remaining cases, which are $n = p2^m$ with $p > 1$ odd and $m \geq 1$.

2. (10 points) For $n \geq 1$ an integer, construct a piecewise linear map $f\colon I \to I$ of a compact interval I such that f has a point of period $2n+1$ but does not have a point of period $2n-1$.

Recall that $\Sigma_m = \{0, 1, \dots, m-1\}^{\mathbb{N}}$ denotes the **shift space** of all sequences $\omega = (\omega_0, \omega_2, \dots)$ with $\omega_k \in \{0, 1, \dots, m-1\}$ for each k. The **left shift map** $\sigma\colon \Sigma_m \to \Sigma_m$ is defined by $\sigma(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \dots)$. For $\lambda > 1$, we put a metric d_λ on Σ_m defined as follows:

$$\text{For } \omega, \omega' \in \Sigma_m, \qquad d_\lambda(\omega, \omega') = \begin{cases} \lambda^{-\min\{k \,|\, \omega_k \neq \omega'_k\}} & \text{if } \omega \neq \omega' \\ 0 & \text{if } \omega = \omega' \end{cases}$$

3. (10 points) Fix some sequence $\omega \in \Sigma_m$ and define its stable set $S(\omega)$ to be the set of all $\alpha \in \Sigma_m$ such that $d_\lambda(\sigma^n(\omega), \sigma^n(\alpha)) \to 0$ as $i \to \infty$. Identify all the sequences in $S(\omega)$.

4. (10 points) We know that the shift map $\sigma\colon \Sigma_m \to \Sigma_m$ is chaotic and therefore has sensitive dependence on initial conditions. Find the **sensitivity constant** for $\sigma\colon (\Sigma_m, d_\lambda) \to (\Sigma_m, d_\lambda)$. That is, find the largest constant $\Delta > 0$ such that for all $\omega \in \Sigma_m$ and $\epsilon > 0$ there exists some $\omega' \in \Sigma_m$ such that $d_\lambda(\omega, \omega') < \epsilon$ and $d_\lambda(\sigma^n(\omega), \sigma^n(\omega')) \geq \Delta$ for some $n \in \mathbb{N}$.

5. Let X and Y be metric spaces and suppose that $f\colon X \to X$, $g\colon Y \to Y$, and $h\colon X \to Y$ are continuous maps such that $h \circ f = g \circ h$, as in the diagram below. Assume also that h is surjective. Here we prove, in particular, that if f is chaotic then g is chaotic.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

   (a) (5 points) Prove that if D is a dense subset of X, then $h(D)$ is a dense subset of Y.

   (b) (5 points) Let $\mathrm{Per}(f) \subset X$ and $\mathrm{Per}(g) \subset Y$ denote the periodic points of f and g respectively. Prove that $h(\mathrm{Per}(f)) \subset \mathrm{Per}(g)$.

   (c) (5 points) Prove that if f is topologically transitive, then g is topologically transitive.

   (d) (5 points) Prove that if f is mixing, then g is mixing.