Math 3890: Dynamical Systems – Assignment 7

Due in-class on Wednesday, March 20

This assignment has 3 questions for a total of 65 points.

For a fixed integer $m \geq 2$, let Σ_m denote the **shift space** $\Sigma_m = \{(\omega_k)_{k=0}^{\infty} \mid \omega_k \in \{0, 1, \ldots, m-1\}\}$ of all infinite sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ of numbers $\omega_k \in \{0, \ldots, m-1\}$. Define the **left shift map** on $\sigma \colon \Sigma_m \to \Sigma_m$ by $\sigma(\omega_0, \omega_1, \omega_2, \ldots) = (\omega_1, \omega_2, \ldots)$, and define a metric d on Σ_m by:

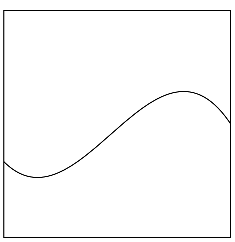
$$\text{For } \omega, \omega' \in \Sigma_m, \qquad d(\omega, \omega') = \begin{cases} 2^{-\min\{k \mid \omega_k \neq \omega'_k\}} & \text{if } \omega \neq \omega' \\ 0 & \text{if } \omega = \omega' \end{cases}$$

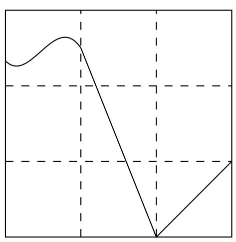
1. Consider the metric space (Σ_m, d) and the shift map $\sigma \colon \Sigma_m \to \Sigma_m$.
   (a) (5 points) Prove that (Σ_m, d) is complete.
   (b) (5 points) Prove that Σ_m is compact. [*Hint:* Use sequential compactness.]
   (c) (5 points) Show that σ is 2–Lipschitz and therefore continuous.
   (d) (5 points) How many fixed points does σ^n have?

2. Let $\Sigma' \subset \Sigma_2$ be the set of all sequences ω satisfying the condition: If $\omega_j = 0$ then $\omega_{j+1} = 1$. That is, Σ' consists of those sequences that never have repeated zeros.
   (a) (5 points) Show that σ preserves Σ' and that Σ' is a closed subset of Σ_2.
   (b) (5 points) Show that periodic points of $\sigma \colon \Sigma' \to \Sigma'$ are dense in Σ'.
   (c) (5 points) Find a point $\omega \in \Sigma'$ whose orbit $\{\sigma^n(\omega) \mid n \in \mathbb{N}\}$ is dense in Σ'.
   (d) (5 points) How many fixed points for σ, σ^2, and σ^3 are there in Σ'?
   (e) (5 points) Find a recursive formula for the number of fixed points of $\sigma^n \colon \Sigma' \to \Sigma'$ in terms of the number of fixed points for σ^{n-1} and σ^{n-2}.

3. Let $I = [0, 1]$ be the unit interval and $f \colon I \to I$ a continuous function. Here we construct a new continuous function whose periods are exactly twice the periods of f. Define the **double of** f to be the function $F \colon I \to I$ with graph (illustrated below) produced as follows: Divide I into thirds, compress the graph of f into the upper left corner of $I \times I$, and fill in the rest of the graph of F as illustrated. Observe that F maps $[0, \frac{1}{3}]$ into $[\frac{2}{3}, 1]$ and vice versa.

graph of f





graph of F

   (a) (5 points) Give a formula for $F(x)$ in terms of f.
   (b) (5 points) Show that F has a unique fixed point p and that $p \in [\frac{1}{3}, \frac{2}{3}]$.
   (c) (5 points) Prove that if $y \in I$ is periodic point for F that is not the unique fixed point, then either y or $F(y)$ lies in $[0, \frac{1}{3}]$ and that, in these respective cases, either $3y$ or $3F(y)$ is a periodic point for f.
   (d) (5 points) Prove that $x \in I$ is a periodic point for f with period n if and only if $x/3$ is periodic for F with period $2n$.