Math 3890: Dynamical Systems – Assignment 6
Due in-class on Wednesday, February 20

This assignment has 6 questions for a total of 60 points.

Recall that we have a continuous projection $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ from the real numbers to the circle $K = \mathbb{R}/\mathbb{Z}$ given by $\pi(x) = [x]$. A *lift* of a map $f\colon K \to K$ is any map $F\colon \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$. The *degree* of a continuous map $f\colon K \to K$ is the number

$$\deg(f) = F(x+1) - F(x) \in \mathbb{Z}$$

where F is any lift and $x \in \mathbb{R}$ is arbitrary.

1. (10 points) Let $f\colon K \to K$ be a continuous map. Suppose that $x_0, y_0 \in \mathbb{R}$ are chosen such that $\pi(y_0) = f(\pi(x_0))$. Prove that there is a unique continuous function $F\colon \mathbb{R} \to \mathbb{R}$ such that $F(x_0) = y_0$ and F is a lift of f (i.e., $\pi \circ F = f \circ \pi$).

2. (10 points) Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ for which there exists a constant $L \geq 0$ such that
$$a_{m+n} \leq a_m + a_n + L \quad \text{for all } m, n \in \mathbb{N}.$$
Prove that $\{\frac{a_n}{n}\}_{n\in\mathbb{N}}$ either converges to a real number $x \in \mathbb{R}$ or else converges to $-\infty$ (meaning that for all K there exists N so that $n \geq N$ implies $\frac{a_n}{n} \leq K$).

3. (10 points) Prove that degree is *multiplicative*: If $f, g\colon K \to K$ are continuous maps, then
$$\deg(f \circ g) = \deg(f)\deg(g).$$

4. (a) (5 points) Is the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + \frac{1}{2}\sin(x)$ the lift of a continuous map of the circle?
   (b) (5 points) Is the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + 4\pi\sin(2\pi x)$ the lift of a circle homeomorphism?

5. For $c, b \in \mathbb{R}$, let $F\colon \mathbb{R} \to \mathbb{R}$ be the map given by $F(x) = x + c + b\sin(2\pi x)$.
   (a) (5 points) Show that if $|2\pi b| < 1$ then F is the lift of an orientation preserving circle homeomorphism f.
   (b) (5 points) Show that if additionally $|c| < |b|$, then f has rotation number $\rho(f) \equiv 0 \mod 1$.

6. (10 points) Let $c_1, c_2 \in \left(0, \frac{1}{2\pi}\right)$. Show that
$$F_1(x) = x + c_1\sin^2(2\pi x), \quad \text{and}$$
$$F_2(x) = x - c_2\cos^2(2\pi x)$$
are lifts of circle maps f_1, f_2 with the same rotation number.