Math 3890: Dynamical Systems – Assignment 5

Due in-class on Wednesday, February 13

This assignment has 6 questions for a total of 60 points.

1. (10 points) Consider metric on the circle $\mathbb{R}/\mathbb{Z}$ given by

$$d([x],[y]) = \min\{|a-b| : a \in [x], b \in [y]\}.$$

   Show that for any α, the rotation $R_\alpha\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $R_\alpha([x]) = [x+\alpha]$ is an isometry (that is $d(R_\alpha([x]), R_\alpha([y])) = d([x],[y])$).

2. (10 points) Given an example of a compact metric space X, a homeomorphism $T\colon X \to X$, and a point $x \in X$ such that $\{T^n(x) \mid n \in \mathbb{Z}\}$ is dense but $\{T^n(x) \mid n \in \mathbb{N}\}$ is not dense.

Recall that a continuous map $f\colon X \to X$ of a compact metric space is **uniquely ergodic** if for every continuous function $\varphi\colon X \to \mathbb{R}$ there is a constant C_f such that $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x))$ converges to C_f for each $x \in X$.

3. (10 points) Show that a contracting map on a compact metric space is uniquely ergodic.

4. (10 points) Give an example of a homeomorphism $T\colon X \to X$ on a compact metric space such that $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(T^i(x))$ converges for every $x \in X$ and continuous function $\varphi\colon X \to \mathbb{R}$, but T is NOT uniquely ergodic.

5. (10 points) What is the frequency of the leading digit of $\{3^n\}$? That is, for $d \in \{1,\ldots,9\}$, find the limit of $\frac{1}{n}\#\{i = 1,\ldots,n \mid d \text{ is the leading digit of } 3^n\}$.

6. (10 points) What is the frequency of the second leading digit of $\{2^n\}$? That is, for $d \in \{0,\ldots,9\}$, find the limit of $\frac{1}{n}\#\{i = 1,\ldots,n \mid d \text{ is the second digit of } 2^n\}$.