Math 3890: Dynamical Systems – Assignment 4
Due in-class on Wednesday, February 6

This assignment has 5 questions for a total of 51 points.

1. Let $f\colon [0,1] \to [0,1]$ be a differentiable map with $|f'(x)| \leq 1$ for all x.
   (a) (7 points) Show that the set of fixed points of f is nonempty and connected (that is, either a point or an interval).
   (b) (7 points) Show that every periodic point of f has period 1 or 2.

For the following problems, let $0 < \alpha < 1$ be irrational and let $T\colon K \to K$ be the associated rotation of the circle $K = \{z \in \mathbb{C} \mid |z| = 1\}$ given by $T(z) = e^{2\pi i\alpha} z$. Alternately, we view the circle K with $\mathbb{R}/\mathbb{Z}$ (the real line with x and $x+1$ identified for). From this perspective, we have a bijection $[0,1) \leftrightarrow K$ (sending $x \in [0,1)$ to $e^{2\pi i x}$) and the rotation T is given by the map

$$T(x) = x + \alpha \mod 1 = (x+\alpha) - \lfloor x + \alpha \rfloor,$$

where $\lfloor y \rfloor \in \mathbb{Z}$ denotes the integer part of $y \in \mathbb{R}$ (so $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$).

2. (10 points) Set $\alpha_0 = \alpha$ and let $c_0 \in \mathbb{N}$ satisfy $c_0\alpha_0 < 1 < (c_0+1)\alpha_0$. Show that $c_0 = \left\lfloor \frac{1}{\alpha_0} \right\rfloor$ and that $T^{c_0+1}(0) \in [0, \alpha)$.

3. (10 points) Denote $\delta_1 = 1 - c_0\alpha_0 > 0$. Let $c_1 \in \mathbb{N}$ satisfy $c_1\delta_1 < \alpha_0 < (c_1+1)\delta_1$. Show that $c_1 = \left\lfloor \frac{1}{\alpha_1} \right\rfloor$, where $\delta_1 = \alpha_1\alpha_0$, and that $0 < \alpha_1 < 1$. Deduce that

$$T^{c_0 c_1 + 1}(0) = T^{c_0(c_1-1)}(T^{c_0+1}(0)) \in [0, \delta_1).$$

4. (10 points) Continuing inductively, given $0 < \alpha_n < 1$, choose $c_n \in \mathbb{N}$ and $0 < \alpha_{n+1} < 1$ such that $1 = \alpha_n(c_n + \alpha_{n+1})$. Show that

$$\alpha = \cfrac{1}{c_0 + \frac{1}{c_1 + \dots}} \qquad \text{(i.e., } (c_0, c_1, \dots) \text{ is the continued fraction expansion of } \alpha)$$

5. (7 points) If α is an irrational number of the form $\alpha = 1/(a_0 + \frac{1}{c})$ where $c > 10^{50}$ and $a_0 \in \mathbb{N}$, then what can one say about the distribution of the orbit $\{T^n(0) \mid n \in \mathbb{N}\}$ of 0? (For example, does it take a short or long time for the orbit to return close to 0?)