Math 3890: Dynamical Systems – Assignment 3
Due in-class on Wednesday, January 30

This assignment has 6 questions for a total of 60 points.

1. (10 points) Prove that a closed subset of a complete metric space is complete (when viewed as a metric space with the restricted metric).

2. (10 points) Let $f\colon X \to Y$ be a map of metric spaces. Prove that f is continuous if and only if one has $f(\overline{A}) \subset \overline{f(A)}$ for every subset $A \subset X$. [Recall that being *continuous* means that the preimage $f^{-1}(U)$ is open in X for every open set U in Y.]

3. (10 points) Suppose $I \subset \mathbb{R}$ is a closed and bounded interval and that $f\colon I \to I$ is such that $|f(x) - f(y)| < |x - y|$ for all $x \neq y$ (this is weaker than the assumption of the Contraction Principle). Prove that f has a unique fixed point $x_0 \in I$ and that $\lim_{n\to\infty} f^n(x) = x_0$ for every $x \in I$.

4. (10 points) Prove that assertion of part (3) is invalid for $I = \mathbb{R}$ by constructing a map $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ for $x \neq y$, but such that f has no fixed point and such that $|f^n(x) - f^n(y)|$ does not converge to 0 for some pair $x, y \in \mathbb{R}$.

5. (10 points) A population of polar lemmings evolves according the the following rules. There are an equal number of males and females. Each lemming lives for two years and dies in the third winter of its life. Each summer, each female lemming produces an offspring of four. In the first summer there is one pair of one-year old lemmings. Let x_n be the total number of lemmings during the nth year. Show that $\frac{x_{n+1}}{x_n}$ converges to a limit $\omega > 1$, and calculate ω.

6. (10 points) Prove that entering any number on a calculator and repeatedly pressing the "`sin`" key gives a sequence that converges to zero. Show that the convergence is exponential when the calculator is set in "degree" mode, but is not exponential when the calculator is set in "radian" mode.