Math 3890: Dynamical Systems – Assignment 2
Due in-class on Wednesday, January 23

This assignment has 3 questions for a total of 63 points.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space X *converges* to a point $x \in X$ if for every open set $U \subset X$ containing x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$, and that the sequence is *Cauchy* if for every $\epsilon > 0$ there exists N such that $d(x_n, x_m) \leq \epsilon$ whenever $m, n \geq N$. A metric space is *complete* if every Cauchy sequence converges (to some point).

1. (10 points) Let ℓ^∞ denote the set of bounded sequences of real numbers. That is, ℓ^∞ is the set of sequences $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ for which there exists some $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Equip ℓ^∞ with the metric

$$d(\{x_n\}, \{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n| .$$

   Prove that (ℓ^∞, d) is a complete metric space.

A metric space is *sequentially compact* if every sequence in it admits a convergent subsequence. In class we proved that compact metric spaces are sequentially compact; the point of the next problem is to prove the converse.

2. Let (X, d) be a sequentially compact metric space.
   (a) (10 points) Prove that for every open cover $\mathcal{U}$ of X, there exists $\delta > 0$ such that any set with with diameter less than δ is contained in some element of $\mathcal{U}$. (The *diameter* of a subset $C \subset X$ is $\operatorname{diam}(C) = \sup\{d(c_1, c_2) \mid c_1, c_2 \in C\}$). Such a number δ is called a *Lebesgue number* for the covering $\mathcal{U}$.
   (b) (10 points) Prove that for any $\epsilon > 0$ there exists a finite covering of X by ϵ–balls. That is: For each $\epsilon > 0$ there is a finite set $F \subset X$ such that $X = \cup_{x\in F} B_\epsilon(x)$.
   (c) (10 points) Use parts (a) and (b) above to prove that X is compact.

The next problem proves that every closed interval $[a, b]$ in $\mathbb{R}$ is compact, which is the key ingredient needed for the Heine–Borel Theorem.

3. Consider $\mathbb{R}$ with the standard Euclidean metric $d(x, y) = |x - y|$, and for $a < b$ consider the closed interval let $I = [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
   (a) (10 points) Prove that every sequence $\{x_n\}$ in $[a, b]$ admits a *monotonic* subsequence. That is, a subsequence $y_j = x_{n_j}$ (where $n_1 < n_2 < \cdots$) such that either $y_j \leq y_{j+1}$ for all j, or else $y_j \geq y_{j+1}$ for all j.
   (b) (10 points) Prove that every monotonic sequence in $[a, b]$ is Cauchy.
   (c) (3 points) Conclude that $[a, b]$ is compact.