Math 3890: Dynamical Systems
Oral Final Exam

Instructions: This class will have oral final exams, in which each student will meet one-on-one with the instructor to present answers to a selection of questions. The oral examinations may be scheduled on any of the three days Tuesday April 23 - Thursday April 25, 2019; each examination will last approximately 30–40 minutes.

The format for the exam is as follows: The student will randomly select to 2 questions from the list of potential questions (below), and then present solutions to these. Student MAY NOT use notes during the exam. During the presentation, the instructor may provide feedback or suggestions if necessary. If you are unable to answer a particular question, of if the presentation finishes early, it maybe possible to answer additional questions.

Questions for the exam:

1. Prove the Contraction Principle

2. For $f\colon K \to K$ an *irrational* rotation of the circle K, prove that for any arc $\Delta \subset K$ one has $\lim_{n\to\infty} \frac{F_\Delta(x,n)}{n} = \ell(\Delta)$, where $F_\Delta(x,n) = \#\{k = 0, \dots, n-1 \mid f^k(x) \in \Delta\}$.

3. Prove that irrational rotations of the circle are uniquely ergodic: That is, for any $0 < \alpha < 1$ irrational and $\varphi\colon K \to \mathbb{R}$ continuous, show that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(R_\alpha^n(x)) = \int_K \varphi(\theta)\, d\theta \qquad \text{for all } x \in K.$$

4. Prove that every orientation-preserving homeomorphism $f\colon K \to K$ of the circle has a well-defined rotation number. That is, show that if $F\colon \mathbb{R} \to \mathbb{R}$ is any lift of f then
$$\rho(F) := \lim_{n\to\infty} \frac{F^n(x)}{n} \quad \text{exists and is independent of } x \in \mathbb{R}.$$

5. Show that an orientation-preserving homeomorphism $f\colon K \to K$ of the circle has a periodic point if and only if its rotation number $\rho(F)$ (of some lift $F\colon \mathbb{R} \to \mathbb{R}$) is rational.

6. Sketch the proof of Poincarè's Theorem: If $f\colon K \to K$ is a minimal, orientation-preserving homeomorphism of the circle, then f is topologically conjugate to the irrational rotation $R_\rho\colon K \to K$ by angle $\rho = \rho(f)$.

7. Sketch the proof for the following fact: If $f\colon K \to K$ is an expanding, orientation-preserving map of the circle with $\deg(f) = 2$, then f is semi-conjugate to the left shift $\sigma\colon \Sigma \to \Sigma$ on the shift space space $\Sigma = \big\{(\omega_k)_{k=0}^\infty \mid \omega_k \in \{0,1\}\big\}$.

8. Sketch the proof for the baby case of Sarkovskii's Theorem: If $f\colon I \to I$ is a continuous self-map of an interval $I \subset \mathbb{R}$ and f has a point of period 3, then f has points of every period $n \in \mathbb{N}$.

9. Let $\sigma_A\colon \Sigma_A \to \Sigma_A$ be the topological Markov chain associated to an aperiodic transition matrix A. Prove that σ_a is topologically mixing and that the periodic points of σ_A are dense in Σ_A.

10. Let $f\colon X \to X$ be continuous map of a compact metric space. Define the topological entropy $h(f)$, prove that it exists and that it is invariant under topological conjugacy, and show that $h(f^m) = mh(f)$ for all $m \in \mathbb{N}$.