Math 9201-01 — Seminar in Topology
Simple curves and the volumes of moduli spaces:
an exploration of the thesis of Maryam Mirzakhani
MWF 1:10–2:00pm in Stevenson Center 1117
Vanderbilt University; Spring 2018

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Office Hours:
MWF 2:00–3:00pm
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Course Information

Course webpage: https://math.vanderbilt.edu/dowdalsd/Sp2018math9201/
The course webpage is the repository for all course information (including this document), announcements, exercises, and the current schedule of topics.

Description: In her celebrated thesis [Mir1] (published in 3 parts in the *Annals*, *Inventiones*, and *JAMS*) Maryam Mirzakhani discovered beautiful connections between (i) the number of simple closed curves of a given length on a hyperbolic surface (ii) the Weil–Petersson volume of the moduli space of hyperbolic surfaces, and (iii) intersection numbers between tautological classes on this moduli space. These discoveries allowed Mirzakhani to:

(1) calculate the precise asymptotic growth rate of the number of simple curves of length at most L on a fixed hyperbolic surface (turns out it is polynomial in L) [Mir4],
(2) find an exact recursion formula relating the volumes of distinct moduli space showing, in particular, that the volume of each moduli space is a polynomial in the length of its boundary [Mir2], and
(3) use the symplectic geometry of the moduli space to read off the intersection numbers of tautological line bundles in terms of these volume polynomials [Mir3]. From this perspective, the volume recursion is shown to satisfy the Virasoro algebra constraints, which leads to a new proof of the celebrated Witten–Kontsevich theorem.

In this seminar course, we will unpack this story with the goal of understanding the statements and proofs of Mirzakhani's results. We will focus mostly on the items (1) and (2) above, but will also attack (3) if time and energy allow.

Additionally, there have been several recent generalizations of Mirzakhani's work that give precise asymptotics for the number of (nonsimple) closed curves of a fixed combinatorial type with length at most L. This was initiated by the contemporaneous work of Mirzakhani [Mir5] and Erlandsson–Souto [ES] in 2016, where the key observation is that the curve counting problem can translated into a problem regarding the convergence of an aptly chosen sequence of measures on the space of geodesic currents. See also Erlandsson–Parlier–Souto [EPS]. This perspective has, in a sense, culminated in the recent work of Rafi and Souto [RS] showing that currents may be used to count a great many things beyond simply curves on surfaces. Time permitting, I hope to cover these more recent developments as well.

Audience: This course is intended for graduate students with interest in topology and geometry, such as hyperbolic geometry, differential and symplectic geometry, and low-dimensional topology. Postdocs and other interested faculty are also welcome to attend.

[*] *This document was updated January 5, 2018. The current version is always available on the course website.*

Principle topics: The central topic will be *hyperbolic surfaces*, that is, 2–dimensional manifolds equipped with a Riemannian metric of constant curvature −1. These will be studied through several lenses including: simple closed curves on a surface, hyperbolic geodesics, the mapping class group, the Teichmüller and moduli spaces of hyperbolic metrics on a surface, the Weil–Petersson symplectic form on the moduli space, and the spaces of measured geodesic laminations and geodesic currents.

Prerequisites: The course will assume solid knowledge of topology (including basic differential and algebraic topology) as well the basics of Riemannian geometry and analysis, including: surfaces and higher-dimensional smooth manifolds, tangent spaces, differential forms, covering spaces, fundamental groups, measure theory and integration. However definitions of essential topics will be reviewed.

Texts: Our principle sources will be Mirzakhani's Thesis [Mir1] and the published papers [Mir2, Mir3, Mir4] resulting from it. We will also be guided by Wolpert's lecture notes [Wol2] on Mirzakhani's work, which focus on the volume recursion and the relation to the Witten–Kontsevich theorem. For the recent advances on counting curves and other quantities with geodesic currents we will consult Rafi–Souto [RS] and the references therein, as this gives the most comprehensive and general account.

We will have to cover a good deal of background before diving in to Mirzakhani's work. Most of the necessary material on hyperbolic surfaces, Teichmüller spaces and mapping class groups can be found in Farb and Margalit's *A primer on mapping class groups* [FM] or Hubbard's *Teichmüller theory* [Hub]. Casson and Bleiler [CB] give a good treatment of measured laminations. There is also Wolpert's book [Wol1] on the basic theory of the Weil–Petersson metric. The original source for the theory of geodesic currents is Bonahon [Bon].

Grading: There will be no graded assignments or exams. Course grades will be assigned based on attendance and participation.

Exercises: There may be occasional exercises to promote individual engagement with the material. The exercises will not be collected, but instead discussed and presented in class.

Presentations: Interested students will be given the opportunity to present selected topics during the semester. If you would like to present some aspect of the course material, please contact me.

Honor Code: Vanderbilt's Honor Code applies to this course.

Tentative Outline:

I. The hyperbolic plane: geodesics, isometries, tangent vectors, geodesic flow, boundary at infinity
II. Hyperbolic surfaces: closed geodesics, covering transformations, the geometry of pairs of pants
III. Mapping class groups: definitions, examples, action on curves
IV. Teichmüller and moduli spaces: Riemann surfaces, uniformization, length functions, Mumford compactness, Fenchel–Nielsen coordinates
V. Measured geodesic laminations and the Thurston measure.
VI. The Weil–Petersson metric: quadratic differentials, the symplectic from, Wolpert's magic formula
VII. The McShane identity, and Mirzakhani's generalized McShane identity for bordered surfaces
VIII. Recursion for Weil–Petersson volumes of moduli spaces: polynomial behavior of WP volume, leading coefficients of the volume polynomials, integration, finding the volumes
IX. Intersection theory and the Witten–Kontsevich Theorem: principle bundles, symplectic reduction, volumes and the Virasoro equations
X. Asymptotic growth of the number of simple closed curves: counting multicurves, counting curves and WP volumes, counting different types of curves.
XI. The space of geodesic currents and the intersection pairing: we're all in this together!
XII. Counting with currents: counting arbitrary MCG-orbits of curves, counting lattice points

References

[Bon] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[CB] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.

[EPS] Viveka Erlandsson, Hugo Parlier, and Juan Souto. Counting curves, and the stable length of currents. Preprint arXiv:1612.05980, 2016.

[ES] Viveka Erlandsson and Juan Souto. Counting curves in hyperbolic surfaces. *Geom. Funct. Anal.*, 26(3):729–777, 2016.

[FM] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Hub] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1.* Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[Mir1] Maryam Mirzakhani. *Simple geodesics on hyperbolic surfaces and the volume of the moduli space of curves.* ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Harvard University.

[Mir2] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, 167(1):179–222, 2007.

[Mir3] Maryam Mirzakhani. Weil-Petersson volumes and intersection theory on the moduli space of curves. *J. Amer. Math. Soc.*, 20(1):1–23, 2007.

[Mir4] Maryam Mirzakhani. Growth of the number of simple closed geodesics on hyperbolic surfaces. *Ann. of Math. (2)*, 168(1):97–125, 2008.

[Mir5] Maryam Mirzakhani. Counting Mapping Class group orbits on hyperbolic surfaces. Preprint arXiv:1601.03342, 2016.

[RS] Kasra Rafi and Juan Souto. Geodesics Currents and Counting Problems. Preprint arXiv:1709.06834, 2017.

[Wol1] Scott A. Wolpert. *Families of Riemann surfaces and Weil-Petersson geometry*, volume 113 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2010.

[Wol2] Scott A. Wolpert. Lectures and notes: Mirzakhani's volume recursion and approach for the Witten–Kontsevich theorem on moduli tautological intersection numbers. Preprint arXiv:1108.0174, 2016.