Geodesic Currents

Apr 16th

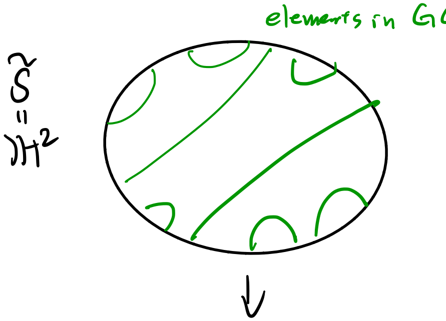
<u>References</u>:

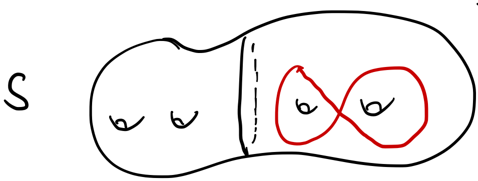
1. the geometry of Teichmüller space via geodesic currents (1998)
2. hyperbolic structure on surfaces and geodesic currents.
3. length function on currents and applications to "dynamics & counting"

1. Bonahon 2. Aramayona & Leininger. 3. Erlandsson & Uyanik.

4. For surface with boundary. Duchin-Leininger-Rafi

Given S hyperbolic, closed, oriented surface.





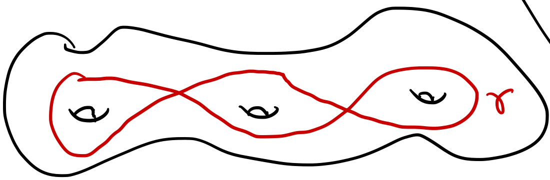
$G(\tilde{S}) = \{$ bi-infinite unoriented complete geodesics in $\partial\mathbb{H}^2 \}$ $= (S^1_\infty \times S^1_\infty \setminus \Delta)/\mathbb{Z}_2$

<u>defn</u>: a <u>geodesic current</u> is a locally finite Borel measure on $G(\tilde{S})$ which is $\pi_1(S)$ invariant.

takes finite values on compact sets

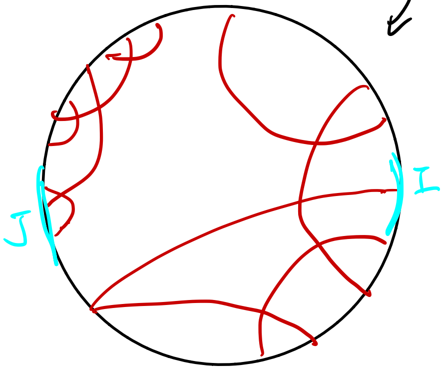
This defn is π_1-invariant.

<u>example</u>:



$p^{-1}(\gamma)$ is a discrete subset of $G(\tilde{S})$

lifts



$\mu_\alpha(I \times J) = \#$ of times a lift of γ in $p^{-1}(\gamma)$ intersect $I \times J$.

<u>e.g.</u> closed curves, lamination on surface.

<u>Fact</u>: the space current is infinite dimensional.

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The space of geodesic currents is equipped with weak $*$ topology: $\mu_n \in \mathrm{Curr}(S)$, $\mu \in \mathrm{Curr}(S)$,

$$\mu_n \to \mu \iff \int f \, d\mu_n \to \int f \, d\mu \quad \forall \text{ cpted supported cont function on } G(\tilde{S})$$

Given α, β closed curves on S, the <u>geometric intersection number</u> $i(\alpha, \beta) = \min \# \{ \alpha' \cap \beta' \mid \forall \alpha' \sim \alpha, \beta' \sim \beta \}$.

$=$ intersection number between geodesic representatives α, β.

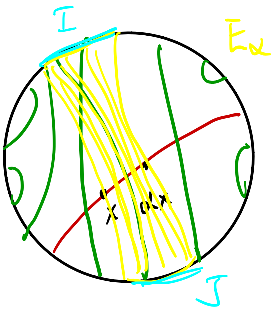
Fact (Bonahon) the multiples of currents from closed curves is dense in $Curr(S)$.

Thm: (Bonahon, 1986). There exist a bilinear homogeneous functional $i: Curr(S) \times Curr(S) \to \mathbb{R}$ s.t. for any two closed curves α, β.

$$i(\mu_\alpha, \mu_\beta) = i(\alpha, \beta).$$

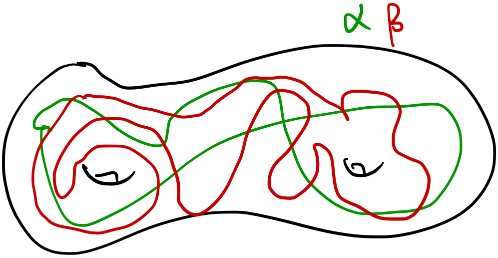
bilinear means $i(a\mu_1 + b\mu_2, \mu_3) = a\, i(\mu_1, \mu_3) + b\, i(\mu_2, \mu_3)$.
homogeneous

In pratice, say $\mu_1 = \mu_\alpha$, $\mu_2 = \mu_\beta$. for some closed curve on S.



$\tilde{\alpha}$ lift of α to $\tilde{S}$, pick fundamental domain in $\tilde{\alpha}$.

Look at all lifts of β passing through $x - \alpha x$, denote E_α with a transverse angle.



$$i(\alpha, \beta) = \mu_\beta(E_\alpha)$$

$$i(\mu_\alpha, \mu) = \mu(I \times J)$$

μ defined by all the geodesics going through $x, \alpha x$.

Define $\mathbb{P}Curr(S) = Curr(S) - \{0\} \big/ \mu \sim a\mu$, $a>0$

<u>Thm</u> (Bonahon) the space of $\mathbb{P}Curr(S)$ is compact.

<u>properties of intersection number</u>:

1. $i(f\mu_1, f\mu_2) = i(\mu_1, \mu_2)$

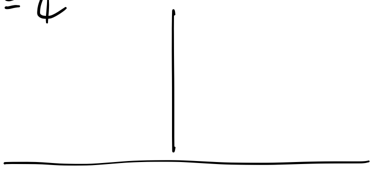
2. $i(\mu, \mu) = 0 \iff \mu \in ML(S)$.

<u>defn</u>: μ is called a <u>filling</u> if $i(\mu, \mu') \neq 0 \ \forall \mu' \in Curr(S)$.

$\boxed{\text{Apr 18th}}$

Goal: embed $T(S)$ into $Curr(S)$ using Liouville currents

$\mathbb{H}^2 \subseteq \mathbb{C}$



<u>Cross-ratio</u>:

$$[z_1, z_2, z_3, z_4] = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

$Isom(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$.

$$z \mapsto \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{R}, \quad ad-bc=1$$

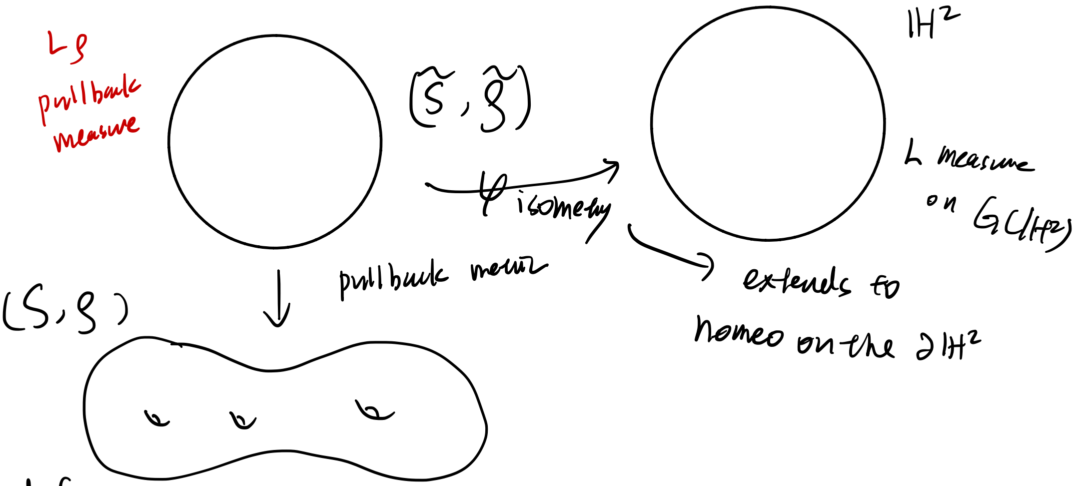
<u>prop</u>: the cross-ratio is invariant under $PSL_2(\mathbb{R})$

<u>Pf</u>: direct substitution $\square$

Liouville measure: Let $[a,b]$ & $[c,d]$ be disjoint intervals in $\mathbb{R} \subseteq \partial \mathbb{H}^2$.

$$L([a,b]\times[c,d]) = \left| \log \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right| \right|$$

* This is indeed Radon measure on $G(\partial \mathbb{H}^2)$
* invariant under $PSL_2(\mathbb{R})$.



defn:
L_g defined on $G(\tilde{S})$ is the pullback of L

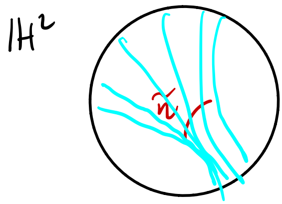
$$L_g(I \times J) = L([a,b]\times[c,d]) \qquad [a,b] = \varphi(I), \quad [c,d] = \varphi(J)$$

* L_g Radon measure, π_1 invariant on $G(\tilde{S})$

second defn:

$\tilde{\eta} : (-\varepsilon, \varepsilon) \to \mathbb{H}^2$ is a geodesic arc unit speed.

$E_{\tilde{\eta}} = \{ \text{geodesics intersect } \tilde{\eta} \text{ transversely} \} \subseteq G(\mathbb{H}^2)$



Notice can parametrize $E_{\tilde{\eta}}$ by 2 variables.

1. pt 2. tangent vector at that pt. (angle)

$\tilde{\eta}(t)$ - where it intersects $\tilde{\eta}$ $(t) \in (-\varepsilon, \varepsilon)$ $\theta \in [0, \pi)$

angle with $\tilde{\eta}'(t)$

$L_{E_{\tilde{\eta}}} = \frac{1}{2} \sin\theta \, d\theta \, dt$.

$$\int_{E_{\tilde{\eta}}} dL = \int_{-\varepsilon}^{\varepsilon} \int_{0}^{\pi} \frac{1}{2} \sin\theta \, d\theta \, dt = 2\varepsilon = \ell(\tilde{\eta})$$

Together, with the description of intersection # from last time, and the 2nd defn. we have the following:

<u>Thm</u> (Bonahon, 80-annals) The map $(X, f) \mapsto L_X$ is a proper embedding of $\mathcal{T}(X) \to \mathrm{Curr}(X)$. For all closed curves, say α, following holds:

$$i(\alpha, L_X) = \ell_X(\alpha)$$

α = μ_α; L_X ↪ Liouville current

$\ell_X(\alpha)$ = length of the geodesic rep of α w.r.t. hyp metric on X.

Note that intersection # $i(\,,\,)$ has a continuous extension to $\mathrm{Curr}(S) \times \mathrm{Curr}(S)$. You can make sense of length of currents w.r.t. X $\ell_X(\mu) = i(\mu, L_X)$

Thm (Otal, 1990 Annals) suppose $\mu_1, \mu_2 \in \mathrm{Curr}(S)$

then $\mu_1 = \mu_2 \iff i(\mu_1, \gamma) = i(\mu_2, \gamma)$ for all

closed curves γ on S.