VIII – Counting with geodesic Currents

Wed 4/25/18

We'll now take a couple days looking at the recent paper "Geodesic Currents & Counting Problems" by Rafi & Souto

point: the measure convergence result $\nu_L^\gamma \to \text{const.}\, \mu_{Th}$ can be generalized to allow you to count more things.

Setup S closed surf,

Teichmüller space $\mathcal{T}(S)$

Mapping class group $\mathrm{Mod}(S)$

Moduli space $\mathcal{M}(S)$

Measured lamination space $\mathcal{ML}(S)$ w/ Thurston meas μ_{Th}

Space of Geodesic currents: $\mathcal{C}(S) = \mathrm{Curr}(S)$

= Space of $\pi_1(S)$ inv't Radon measures on $\partial^2 \mathbb{H}^2$, $\mathbb{H}^2 = \tilde{S}$

– equip w/ weak-$*$ topology

Recall from Caglar's Lectures:

- $\mathcal{ML}(S)$ & $\mathcal{T}(S)$ embed in $\mathcal{C}(S)$ ($X \in \mathcal{T}(S) \rightsquigarrow$ Liouville current L_X)
- $\exists$ continuous intersection pairing $i: \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+$
  - homogeneous in both entries
  - if α, β curves, $X \in \mathcal{T}(S)$ metric
    - $i(\alpha, \beta) =$ geom int #
    - $i(\alpha, X) = \ell_\alpha(X)$ hyp length
    - $i(X, X) = \pi^2 |\chi(S)|$
  - $\mathcal{ML}(S) = \{\alpha \in \mathcal{C}(S) \mid i(\alpha, \alpha) = 0\}$

Def a current $\lambda \in \mathcal{C}(S)$ is <u>filling</u> if every geodesic in S is transversely intersected by some geod in $\mathrm{supp}(\lambda)$.

Ex If γ is a filling multicurve (i.e. intersects every scc), then γ is a filling current.

· If $X \in \mathcal{T}(S)$, then the Liouville current L_X is filling.

Prop: If λ any filling current, then
$$\{\mu \in \mathcal{C}(S) \mid i(\lambda, \mu) \leq 1\} \text{ is compact in } \mathcal{C}(S).$$

Notation $f: \mathcal{C}(S) \to \mathbb{R}_+$ positive, cont, homog, set
$$m(f) = \mu_{Th}\{\lambda \in \mathcal{ML} \mid f(\lambda) \leq 1\}$$
(set is compact so has finite measure)

Ex: α filling curve $\rightsquigarrow f(\cdot) = i(\alpha, \cdot)$, get
$$m(\alpha) = \mu_{Th}\{\lambda \in \mathcal{ML} \mid i(\alpha, \lambda) \leq 1\}$$

· $X \in \mathcal{T}(S)$, then $f(\cdot) = i(X, \cdot)$, so
$$m(X) = \mu_{Th}\{\lambda \in \mathcal{ML} \mid \ell_\lambda(X) \leq 1\} = B(X)$$
from before!

set $m_g = \int m(X)\, dVol_{wp}(X)$ $(= b_{g,n}$ from before$)$

Thm (Rafi-Souto)

$f: \mathcal{C}(S) \to \mathbb{R}_+$ positive, cont, homog, & $\alpha \in \mathcal{C}(S)$ filling current.
$$\lim_{L \to \infty} \frac{\#\{\phi \in \mathrm{Mod}(S) \mid f(\phi(\alpha)) \leq L\}}{L^{6g-6}} = \frac{m(\alpha)\, m(f)}{m_g}.$$

Related Thm (Mirzakhani '16)

for any closed curve γ on S (simple, filling, nonfilling, etc),
$\exists$ const c_γ s.t. $\forall$ hyp metric $X \in \mathcal{T}$, have

$$\frac{\#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \mid \ell_\gamma(X) \leq L\}}{L^{6g-6}} \to c_\gamma \frac{B(X)}{b_{g,n}}$$

(So, there are 2 different strengthenings of Mirzakhani's original result)

- Erlandson: consider $\pi_1(S,p)$ (with "simple generating set"). Then there exists a current λ_S s.t. $\forall \gamma \in \pi_1(S,p)$ its conjugacy length is $\|\gamma\| = i(\gamma, \lambda_S)$. Hence $\|\cdot\| = i(\cdot, \lambda_S)$ extends to a function on $\mathcal{C}(S)$, and get $\frac{\#\{\phi \in \mathrm{Mod} \mid \|\phi(\gamma)\| \leq L\}}{L^{6g-6}} \to \frac{m(\gamma)m(\|\cdot\|)}{m_g}$.

Cool Application of Rafi-Souto to Lattice Point Counting:

consider Teichmüller space $\mathcal{T}(S)$.

equip with Thurston's Lipschitz metric:

$(X,f), (Y,g) \in \mathcal{T}(S)$,

$$d_{Thu}(X,Y) = \inf\left\{\log(\mathrm{Lip}\,\phi) \;\middle|\; \begin{array}{l}\phi \simeq g\circ f^{-1} : X \to Y \\ \text{Lipschitz}\end{array}\right\}$$

$$\begin{tikzcd} & X \arrow[dd, dotted, "\phi"] \\ S \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\simeq"] & \\ & Y \end{tikzcd}$$

for $X \in \mathcal{T}(S)$, def $D_X : \mathcal{C}(S) \to \mathbb{R}_+$, $D_X(\eta) = \max_{\mu \in \mathcal{ML}} \frac{i(\eta,\mu)}{\ell_\mu(X)}$

continuous, positive, homogeneous

(Thurston) $d_{Thu}(X,Y) = \log D_X(Y)$ $\qquad e^R = L$

Apply Rafi-Souto to $f = D_X$ $\alpha = Y$, with $R = \log(L)$
$\forall X, Y \in \mathcal{T}(S)$:

get $$\lim_{R\to\infty} \frac{\#\{\phi \in \mathrm{Mod}(S) \mid d_{Thu}(X, \phi Y) \leq R\}}{e^{(6g-6)R}} = \frac{m(D_X)\,m(Y)}{m_g}$$

$$\left(d_{Th}(Y, \phi(Y)) \leq R \iff D_X(\phi(Y)) \leq L\right)$$

Key to proving main thm is again measure convergence result.

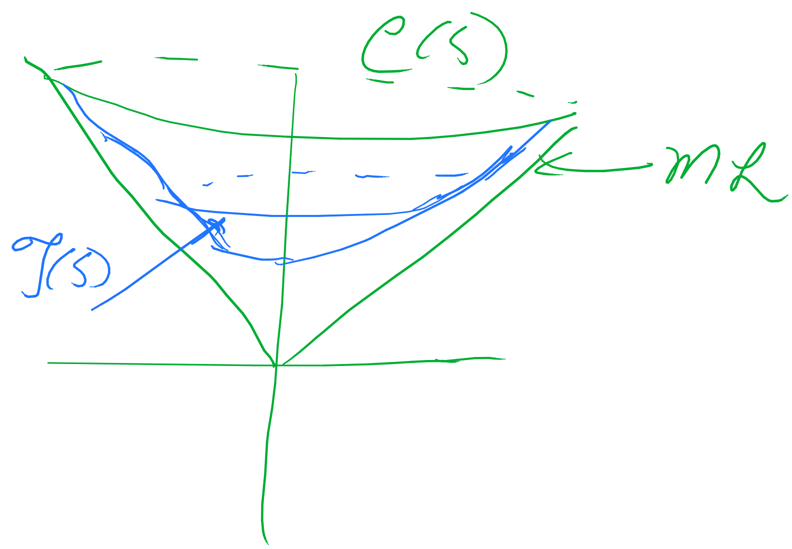
Measure Convergence Thm (MCT) (Rafi–Souto)

for $\alpha \in \mathcal{C}(S)$ any filling current. Set $\underbrace{\nu_L^\alpha = \frac{1}{L^{6g-6}} \sum_{\phi \in \mathrm{Mod}(S)} \delta_{\frac{\phi(\alpha)}{L}}}_{\text{a measure on } \mathcal{C}(S)}$

Then ν_L^α converges to $\frac{m(\alpha)}{m_g} \mu_{Th}$, in weak $*$ topology as $L \to \infty$.

– in particular, supported on $\mathcal{ML} \subset \mathcal{C}$!

In Curr; if you look at any Mod orbit & rescale; it accumulates & equidistributes on $\mathcal{ML}$!



Proof of Thm assuming MCT:

$$\frac{\#\{\phi \in \mathrm{Mod}(S) \mid f(\phi(\alpha)) \le L\}}{L^{6g-6}} = \frac{1}{L^{6g-6}} \Big(\sum_{\phi \in \mathrm{Mod}} \delta_{\phi(\alpha)} \Big) \big(\{ \eta \in \mathcal{C}(S) \mid f(\eta) \le L \} \big)$$

$$= \frac{1}{L^{6g-6}} \Big(\sum_{\phi \in \mathrm{Mod}} \delta_{\frac{\phi(\alpha)}{L}} \Big) \big(\{ \eta \in \mathcal{C}(S) \mid f(\eta) \le 1 \} \big)$$

$$= \nu_L^\alpha \big(\{ \eta \in \mathcal{C}(S) \mid f(\eta) \le 1 \} \big)$$

$$\longrightarrow \frac{m(\alpha)}{m_g} \mu_{Th} \big(\{ \eta \in \mathcal{C}(S) \mid f(\eta) \le 1 \} \big) = \frac{m(\alpha)\, m(f)}{m_g}. \quad \blacksquare$$

So: The MCT is key to everything.

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In her thesis, Mirzakhani deduces MCT from her results on the volume of moduli spaces

(the polynomial nature of volume + insight to integrate $s_X(\gamma, L)$ over Moduli space + swap order of integrating and taking limit using Lebesgue dominated convergence)

In her more recent 2016 paper, Mirzakhani instead counts a specific thing directly using a different method.

Rafi-Souto take same approach + show following:

Thm 2 (R-S; a more precise version of Mirzakhani count)

X hyp surf, γ filling closed curve. Then

$$\frac{\#\{\phi \in Mod(S) \mid l_X(\phi(\gamma)) \leq L\}}{L^{6g-6}} \longrightarrow \frac{m(X)\, m(\gamma)}{m_g}$$

Rmk · Special Case of main Thm:

· use it to prove MCT, + then deduce main thm in general.

Idea to prove MCT:

1) argue (similar as to before) that $\{\nu_L^\alpha\}$ is precompact in space of measures on $\mathcal{C}(S)$, + that every accumulation point is locally finite and positive

fix ref pt $X \in \mathcal{T}(S)$ + show $\forall T > 0$ that

$$\sup_L \nu_L^\alpha \{\lambda \in \mathcal{C}(S) \mid l_X(\lambda) \leq T\} < \infty$$

2) any accumulation pt ν (= subsequential limit of ν_L^α) is supported on $\mathcal{ML}(S) \subset \mathcal{C}(S)$.

pf: Since $\mathcal{ML} = \{\lambda \in \mathcal{C} \mid i(\lambda,\lambda)=0\}$, suffices to show:

$T>0$, $$\int_{\{\lambda\in\mathcal{C} \mid l_x(\lambda)\le T\}} i(\lambda,\lambda)\, d\nu(\lambda) = 0$$

$x \in \mathcal{T}(S)$ fixed ref surf

Compute for ν_L^α + take limit:

$$\int_{\{\lambda\in\mathcal{C} \mid l_x(\lambda)\le T\}} i(\lambda,\lambda)\, \nu_L^\alpha(\lambda) = \frac{1}{L^{6g-6}} \sum_{\substack{\phi\in \mathrm{Mod},\\ l_x\left(\frac{\phi(\alpha)}{L}\right)\le T}} i\left(\frac{\phi(\alpha)}{L}, \frac{\phi(\alpha)}{L}\right)$$

$$\overset{\text{invariance}}{=} \frac{1}{L^{6g-6}} \sum_{\substack{\phi\in \mathrm{Mod}\\ l_x\left(\frac{\phi(\alpha)}{L}\right)\le T}} \frac{i(\alpha,\alpha)}{L^2}$$

$$= \int_{\{\lambda\in\mathcal{C} \mid l_x(\lambda)\le T\}} \frac{i(\alpha,\alpha)}{L^2}\, \nu_L^\alpha(\lambda) = \frac{i(\alpha,\alpha)}{L^2}\, \underbrace{\nu_L^\alpha(\{\lambda\in\mathcal{C} \mid l_x(\lambda)\le T\})}_{\sup < \infty \text{ by above}}$$

$\to 0$ ∎

3) if ν an accumulation pt of $\{\nu_L^\alpha\}$ then for any simple curve γ,

$$\nu(\{\lambda\in \mathcal{ML} \mid i(\lambda,\gamma)=0\}) = 0$$

(similar sorts of estimates as in (1))

4) (Mirzakhani-Lindenstrauss): Scalar multiples $c\mu_{Th}$ of Thurston measure are the only locally finite, Mod-invariant measures on $\mathcal{ML}$ satisfying (3) above.

$\Rightarrow$ any accumulation pt ν of $\{\nu_L^\alpha\}$ has form $\nu = c\cdot\mu_{Th}$

Remains to calculate the constant:

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5) any accumulation point ν of $\{\nu_L^\alpha\}$ has form $\nu = \frac{m(\alpha)}{m_g}\mu_{Th}$

pf: Choose $L_n \to \infty$ st $\nu = \lim_{n\to\infty} \nu^\alpha_{L_n}$. write $\nu = c \cdot \mu_{Th}$

Use Thm 2: fix ref surf $X \in \mathcal{T}_g$, a filling curve γ.

have seq of measures $\nu^\gamma_{L_n}$ as well.

Pass to further subseq st $\nu^\gamma_{L_n} \to h \cdot \mu_{Th}$.

Recall $f : \mathcal{C} \to \mathbb{R}_+$ cont, homog, + β filling, then

$$\frac{\#\{\phi \in \mathrm{Mod} \,/\, f(\phi(\beta)) \le L\}}{L^{6g-6}} = \nu_L^\beta(\{\eta \in \mathcal{C} \,|\, f(\eta) \le 1\})$$

as $n \to \infty$

$$\frac{\#\{\phi \,|\, i(\alpha, \phi(\gamma)) \le L_n\}}{L_n^{6g-6}} = \nu^\gamma_{L_n}(\{\eta \,|\, i(\alpha,\eta) \le 1\}) \to h\, m(\alpha)$$

$$\|$$

$$\frac{\#\{\phi \,|\, i(\phi(\alpha), \gamma) \le L_n\}}{L_n^{6g-6}} = \nu^\alpha_{L_n}(\{\eta \,|\, i(\gamma,\eta) \le 1\}) \to c \cdot m(\gamma)$$

also

$$\frac{\#\{\phi \,|\, i(X, \phi(\gamma)) \le L_n\}}{L_n^{6g-6}} = \nu^\gamma_{L_n}(\{\eta \,|\, i(X,\eta) \le 1\}) \to h\, m(X)$$

$\xrightarrow{\text{Thm 2}} \frac{m(X)\, m(\gamma)}{m_g}$

so $$\frac{\#\{\phi \,|\, i(X, \phi(\gamma)) \le L_n\}}{\#\{\phi \,|\, i(\alpha, \phi(\gamma)) \le L_n\}} \to \frac{m(X)}{m(\alpha)}$$

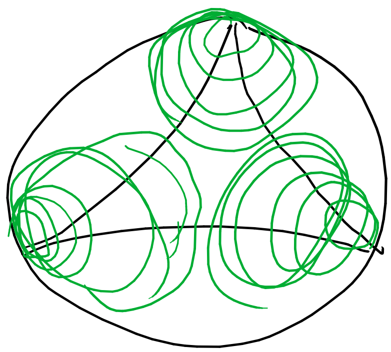
$$\Rightarrow \frac{\#\{\phi \,|\, i(\alpha, \phi(\gamma)) \le L_n\}}{L_n^{6g-6}} \to \frac{m(\alpha)}{m(X)}\left(\frac{m(X)\, m(\gamma)}{m_g}\right),$$ so $\boxed{c = \frac{m(\alpha)}{m_g}}$

Remains to prove Thm 2 (Sketch):

- $\mathcal{ML}$ a PL mnfld. In fact comes with a natural symplectic structure whose volume form gives Thurston measure μ_{Th}.
- Let λ be a geodesic lamination whose complementary regions are ideal triangles. Set $\mathcal{ML}(\lambda)$ = measured laminations that are transverse to λ.

  (open dense set in $\mathcal{ML}$, with full μ_{Th}-measure)
- Thurston introduced a global parametrization $\Psi_\lambda : \mathcal{T}(S) \to \mathcal{ML}(\lambda)$ as follows: Given $X \in \mathcal{T}(S)$. Realize λ geodesically.

  each component of $X \setminus \lambda$ is an ideal geodesic triangle



- give triangle horocycle foliation

  (leaves are intersection of horocycles based at ideal endpts)
- Choose maximal horocycles that mutually touch.

  (This foliation of the geod triangle is unique; determined by hyperbolic structure).

Use these horocycles as leaves on each triangle. Leaves on adjacent triangles piece up

$\rightsquigarrow$ get a bunch of leaves on X that can pull tight to geodesic lamination μ.

For the transverse measure to μ, use geodesic length of λ!

(length of an arc $\pitchfork$ to μ is basically hyp length of corresp. arc of λ)

This defines a measured lamination $\Psi_\lambda(X) \in \mathcal{ML}(\lambda)$

Amazing construction of Thurston: process can be reversed (given $\mu \in \mathcal{ML}(\lambda)$, build hyp metric X st $\Psi_\lambda(X) = \mu$), and Ψ_λ is a homeo.

- (Bonahon–Sözen) $\Psi_\lambda : \mathcal{T}(S) \to \mathcal{ML}(\lambda)$ is a symplectomorphism between $\mathcal{T}(S)$ with Weil-Petersson symplectic form & $\mathcal{ML}(\lambda) \subset \mathcal{ML}$ with Thurston's symplectic form.

Thus $\boxed{\Psi_\lambda \text{ preserves volume}}$

- Idea: relate counting problem in Thm 2 to Weil-Petersson volumes of sets $B_{\mathcal{T}}(\gamma, L) = \{X \in \mathcal{T}(S) \mid \ell_\gamma(X) \leq L\} \subseteq \mathcal{T}(S)$

- <u>Thm</u> $\displaystyle \lim_{L \to \infty} \frac{\mathrm{Vol}_{wp}(B_{\mathcal{T}}(\gamma, L))}{L^{6g-6}} = m(\gamma)$

pf: pick some λ & set $B_L(\Psi_\lambda(B_{\mathcal{T}}(\gamma,L)))$; (closed, convex in $\mathcal{ML}$)

Then $\mathrm{Vol}_{wp}(B_{\mathcal{T}}(\gamma, L)) = \mu_{Th}(B_L)$

In $\mathcal{ML}$, can use PL-structure:

$$\mu_{Th}(B_L) = L^{6g-6} \mu_{Th}(\tfrac{1}{L} B_L) \qquad \{\tfrac{1}{L}\eta \mid \eta \in B_L\}$$

<u>Lem</u> If $\{X_n\}$ seq in $\mathcal{T}(S)$ & $L_n > 0$ are such that $\frac{1}{L_n} X_n \to \eta$ in $\mathcal{C}(S)$, then $\frac{1}{L_n}\Psi_\lambda(X_n) \to \eta$ also

Hence: sets $\frac{1}{L} B_L \xrightarrow{\text{ptwise}} \{\eta \in \mathcal{ML}(\lambda) \mid i(\gamma, \eta) \leq 1\}$. So

$$\frac{1}{L^{6g-6}} \mathrm{Vol}_{wp}(B_{\mathcal{T}}(\gamma, L)) = \frac{1}{L^{6g-6}} \mu_{Th}(B_L) = \mu_{Th}(\tfrac{1}{L} B_L)$$

$$\to \mu_{Th}(\{\eta \in \mathcal{ML}(\lambda) \mid i(\gamma, \eta) \leq 1\})$$

$$= \mu_{Th}(\{\eta \in \mathcal{ML} \mid i(\gamma, \eta) \leq 1\}) = m(\gamma)$$

since $\mathcal{ML}(\lambda)$ has full measure ☒

Final ingredient: Lattice pt counting result

(Mirzakhani)

$$\#\{\varphi \in \mathrm{Mod} \mid \ell_{\varphi(\gamma)}(X) \leq L\}$$

$$= \{\varphi \in \mathrm{Mod} \mid \ell_{\gamma}(\varphi^{-1}X) \leq L\} \sim \mathrm{Vol}_{wp}\left(B_g(\gamma, L)\right) \frac{m(X)}{m_g}$$

Thus we get:

$$\frac{\#\{\phi \in \mathrm{Mod} \mid \ell_{\phi(\gamma)}(X) \leq L\}}{L^{6g-4}} \sim \frac{m(X)}{m_g} \frac{\mathrm{Vol}_{wp}\left(B_g(\gamma, L)\right)}{L^{6g-6}}$$

$$\sim \frac{m(X)\, m(\gamma)}{m_g} \qquad \blacksquare$$