VII - Growth of Simple closed curves

We now use the theory of measured laminations we've developed, plus the volume polynomials, to calculate the asymptotic growth rate of # simple closed curves.

Goal: Thm (Mirzakhani)

Let X be a hyp surf $\cong S_{g,n}$

$s_X(L) = \#$ sccs on X of length $\leq L$. Then

$$\lim_{L \to \infty} \frac{s_X(L)}{L^{6g-6+2n}} = n_X \text{ exists.}$$

further: relate n_X to geom of X.

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fix S surf genus g, n boundary components.

Recall: each $X \in \mathcal{T}_{g,n}(L)$, $+ \mu \in \mathcal{ML}(S)$

get length $\underline{l_\mu(X)}$.

<u>Def</u> for $X \in \mathcal{T}_{g,n}(L)$, set

$$B_X = \{\lambda \in \mathcal{ML}(S) \mid l_\lambda(X) \leq 1\} \subset \mathcal{ML}(S)$$

– unit ball for length funct $l_{\cdot}(X)$

Define function $B : \mathcal{T}_{g,n}(L) \to \mathbb{R}$

$$\boxed{B(X) = \mu_{Th}(B_X)}$$

Counting Multicurves

$S = S_{g,n}$

integral multicurve $\gamma = \sum_{i=1}^{n} c_i \gamma_i$ where

$\gamma_1, \dots, \gamma_n$ disjoint essential sccs, $c_i \in \mathbb{N}$.

Let $\mathcal{ML}(S;\mathbb{Z}) = \mathcal{ML}_{g,n}(\mathbb{Z})$ be set of integral multicurves

$\subset \mathcal{ML}(S)$

(recall, sccs embed into $\mathcal{ML}(S)$)

Fact: for train track chart $\Phi_\tau : W_\tau \to V(\tau) \subset \mathcal{ML}(S)$, $W_\tau \subset \mathbb{R}^{B_\tau}$

have. $W_\tau \cap \mathbb{Z}^{B_\tau} \xleftrightarrow{\Phi_\tau} V(\tau) \cap \mathcal{ML}(S;\mathbb{Z})$

(pf: pretty clear: a multicurve carried on τ must give integral weights, + integral weight fn must determine an integral multicurve)

For $X \in \mathcal{T}(S)$, $L > 0$, def

length as a lamination, e.g.

$$b_X(L) = \#\{\gamma \in \mathcal{ML}(S;\mathbb{Z}) \mid \ell_\gamma(X) \le L\}$$

so: $b_X(L) = \#(B(X) \cap \mathcal{ML}(\mathbb{Z}))$

For X fixed, asymptotic behavior of $b_X(L)$ governed by $B(X)$:

Thm for $X \in \mathcal{T}(S)$, $\dfrac{b_X(L)}{L^{6g-6+2n}} \longrightarrow B(X)$

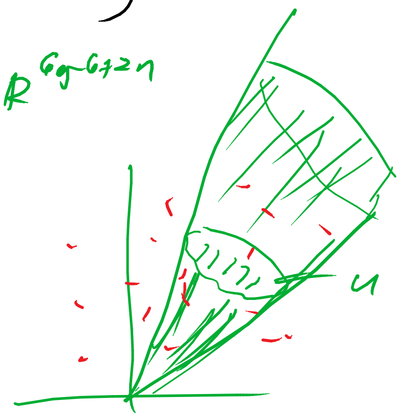
proof: for τ maximal, recurrent track,

for $U \subset \mathcal{ML}(S)$, set

$$b_\tau(U, L) = \#\Big(\underbrace{\mathcal{ML}(S;\mathbb{Z}) \cap V(\tau)}_{\text{integral pts in } W_\tau \subset \mathbb{R}^{6g-6+2n}} \cap \overbrace{L\cdot U}^{=\{\lambda x \,\mid\, x \in U,\ 0 \le \lambda \le L\}}\Big)$$

Thus usual lattice pt counting $\Rightarrow$

$$\frac{b_\tau(U, L)}{L^{6g-6+2n}} \to \mu_{Th}\big(V(\tau) \cap U\big)$$



Cover $\mathcal{ML}(S)$ by finitely many train track charts.

in each we get correct asymptotics, for

$$\frac{b_\tau(B(X), L)}{L^{6g-6+2n}},$$

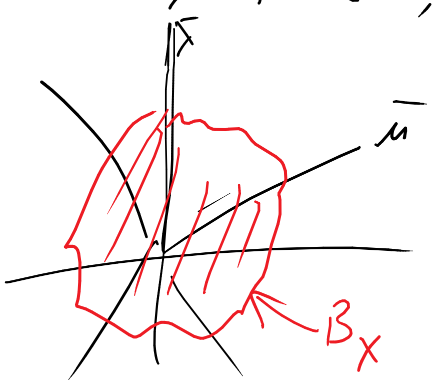
so get same asymptotics overall

(inclusion/exclusion principle: partition $B(X)$ according to nerve of cover by TT charts) ∎

Prop: function $B : \mathcal{T}(S) \to \mathbb{R}_+$ is continuous

proof: know $\ell_\lambda : \mathcal{T}(S) \to \mathbb{R}$ smooth $\forall \lambda \in \mathcal{ML}(S)$

each $\bar{\lambda} \in \mathcal{PML}(S)$, get $\mathbb{R}_+$-line $\mathbb{R}_+\lambda \subset \mathcal{ML}(S)$



Hence, as X varies,

$(\partial B_X) \cap \mathbb{R}_+\lambda$ varies smoothly along line $\mathbb{R}_+\lambda$

Thus: $X \to X' \Rightarrow$

$B_X \to B_{X'}$ (say in Hausdorff top)

Thus $X \mapsto B(X) = \mu_{Th}(B_X)$ is cont ∎

Notice: $B : \mathcal{T}(S) \to \mathbb{R}_+$ descends to

$$B : \mathcal{M}(S) \to \mathbb{R}_+$$

Next goal: B is proper + integrable on $\mathcal{M}(S)$

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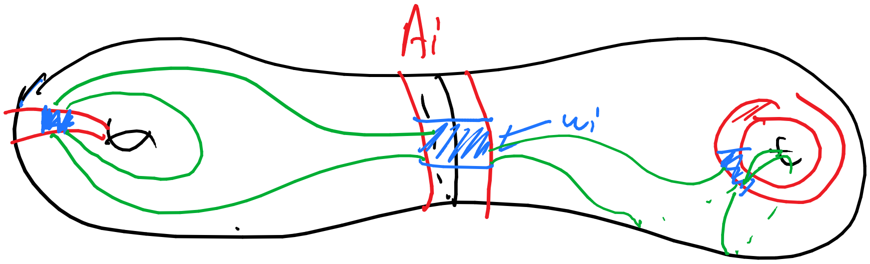
To do this, use

Dehn Coordinates on $\mathcal{ML}(S;\mathbb{Z})$:

Let $P = \{\alpha_1, \dots, \alpha_{3g-3+n}\}$ be a pants decomp of S
around each α_i take an embedded annulus

$A_i \cong S^1 \times [-1,1] \subset S$, with a

fixed "window" $w_i = \text{arc} \times [-1,1] \subset A_i$



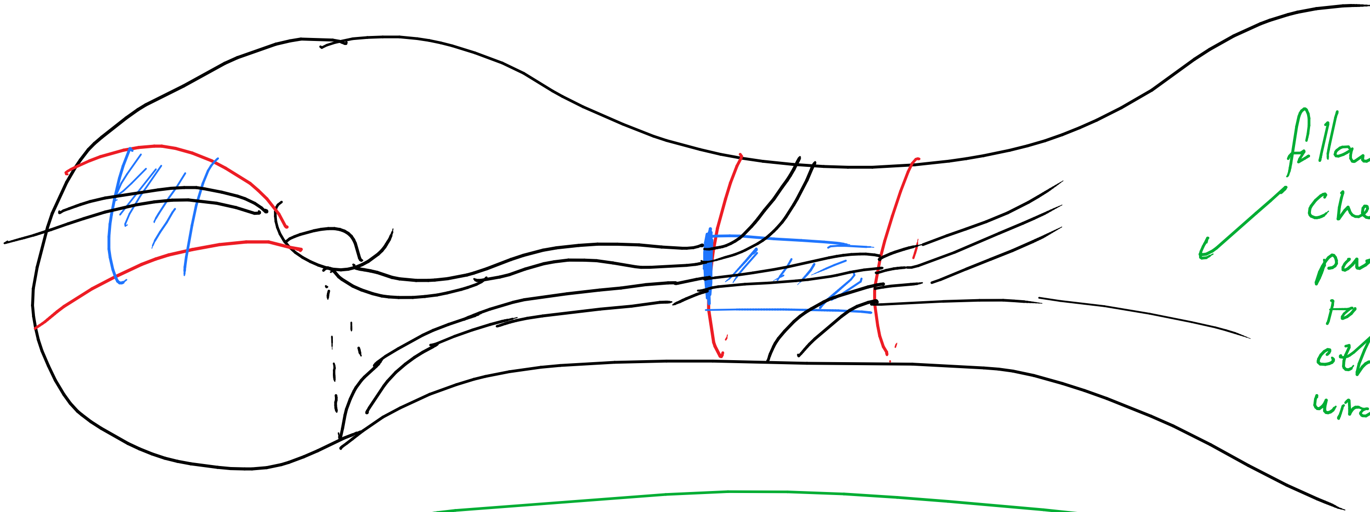
Fix arcs connecting each pair of windows in each pants

Let $C \in \mathcal{ML}(S;\mathbb{Z})$ be any multiple curve,

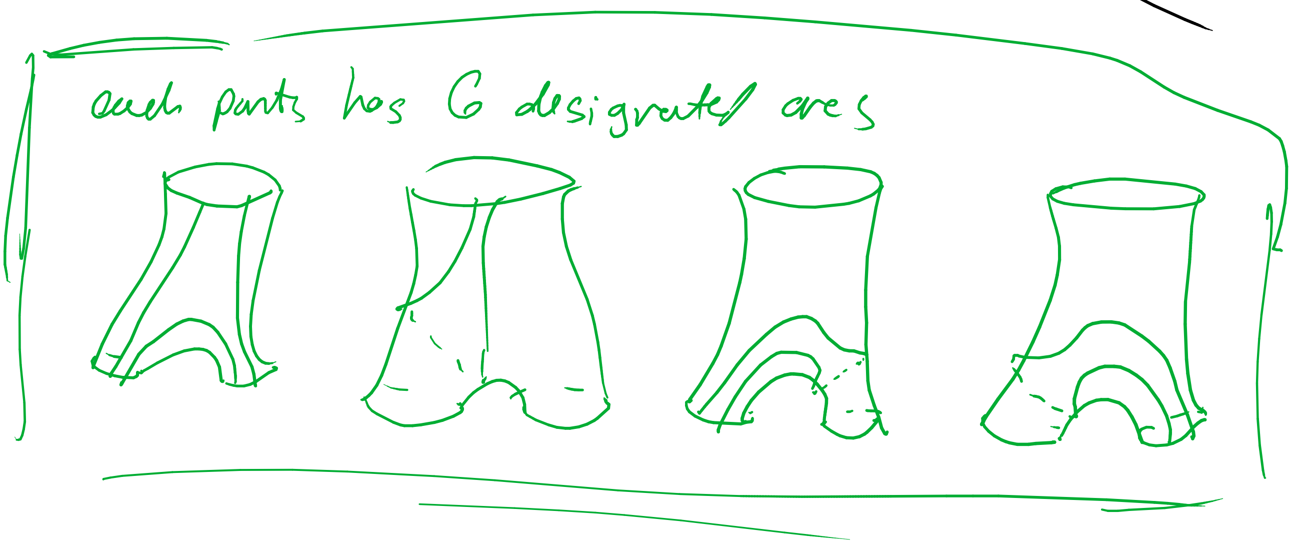
realize as 1-mfld $C \subset S$ (so may have parallel curves)
up to htpy, realize $C \subset S$ s.t.

- $C \pitchfork \partial A_i$ with $C \cap \partial A_i \subset w_i \times \{-1,1\}$
- all components of C htpic to α_i lie in Annulus A_i

· $C \cap$ (any pair of pants) consists of arcs that are homotopic, rel endpts in $U\, w_i \times \{-1,1\}$, to our chosen arcs



follow chosen paths to other windows



define parameters: $m_i = i(C, \alpha_i) \in \mathbb{Z}_{\geq 0}$

· twist $t_i = \begin{cases} \cdot \ \text{\# strands of } C \text{ wrapping left}(+)/\text{right}(-) \text{ around } A_i, & m_i > 0 \\ \cdot \ \text{\# components of } C \text{ parallel to } \alpha_i, & m_i = 0 \end{cases} \in \mathbb{Z}$

<u>Fact</u> Given multiple curve C, effectively unique way to homotope C to satisfy these conditions

$\Rightarrow$ parameters m_i & t_i well defined.

Thm (Dehn–Thurston) fixing pants decomp $P = \{\alpha_1, \dots, \alpha_{3g-3+n}\}$ & windows & chosen arcs in each pants, the map

$$\mathcal{ML}(S;\mathbb{Z}) \longrightarrow \mathbb{Z}^{6g-6+2n}$$
$$C \longmapsto \text{parameters } m_i$$

gives a bijection between $\mathcal{ML}(S;\mathbb{Z})$ & subset

$Z(P)$ of $(m_1, t_1, \dots, m_k, t_k) \in \mathbb{Z}^{2k}$ s.t.

- $m_i \geq 0 \quad \forall i$
- $m_i = 0 \Rightarrow t_i \geq 0$
- $\alpha_i, \alpha_j, \alpha_k$ bound a pants $\Rightarrow m_i + m_j + m_k \in 2\mathbb{Z}$.

proof:
- parameters clearly satisfy these conditions,
- given any parameters, unique way to match up arcs to get a multicurve. ☒

Combinatorial Length Fix Dehn–coords as above,

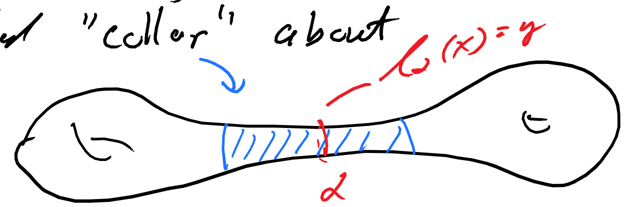
given $\gamma \in \mathcal{ML}(S;\mathbb{Z})$ & $X \in \mathcal{T}(S)$,

Def: combinatorial length γ at X:

$$L(X,\gamma) = \sum_{i=1}^{k} \left(m_i\, S(l_{\alpha_i}(X)) + |t_i|\, l_{\alpha_i}(X) \right),$$

where $S(y) = \operatorname{arcsinh}\left(\dfrac{1}{\sinh(y/2)} \right)$

= width of an embedded "collar" about geod of len y.



Thm Fix Pants $P = \{\alpha_i\}$ + Dehn coords.

given R, $\exists$ const $C(R) \geq 1$ s.t.

If $X \in \mathcal{T}(S)$ has $l_{\alpha_i}(X) \leq R$ $\forall i = 1, \dots, k$,

then for every multiple curve $\gamma \in \mathcal{ML}(S; \mathbb{Z})$, have

$$\boxed{\frac{1}{C(R)} L(X, \gamma) \leq l_\gamma(X) \leq C(R)\, L(X, \gamma)}$$

proof sketch: just hyperbolic geometry:

given γ, realize as a geodesic. Then adjust to a "broken geodesic" that spirals around pants curves + jumps between pants curves by taking the shortest "seam" arcs connecting them.

Point is: $\cdot$ const $C = C(L) > 0$ s.t. if $l_\alpha(X) \leq L$ $\forall \alpha \in P$, then $\frac{1}{C} L(X, \gamma) \leq \text{length}(\text{broken geod for } \gamma) \leq C\, L(X, \gamma)$

(basically, broken geod is described by combinatorics of Dehn coordinates! and you can just write down its length).

- Clear $l_\gamma(X) \leq \text{len}(\text{broken geodesic})$
- hyp geometry of triangles etc $\Rightarrow$

  $l_\gamma(X) \geq \text{const} \cdot \text{len}(\text{broken geodesic})$ ∎

Now, want to use this to show B props + integrable on $\mathcal{M}(S)$.

Def $R : \mathbb{R}_+ \to \mathbb{R}_+$

$$R(x) = \frac{1}{x|\log x|}$$

Thm given suff small $\varepsilon > 0$, $\exists\, C_1, C_2 > 0$ (dep on ε & g,n ($S \cong S_{g,n}$)) s.t. $\forall X \in \mathcal{T}(S)$,

$$B(X) \geq C_1 \prod_{\gamma : l_\gamma(X) \leq \varepsilon} R(l_\gamma(X)), \quad \text{and}$$

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$$\frac{b_X(L)}{L^{6g-6+2n}} \leq C_2 \prod_{\gamma : l_\gamma(X) \leq \varepsilon} \left(R(l_\gamma(X)) + \frac{1}{l_\gamma(X)} \right) \quad \text{for all large } L.$$

proof sketch:

use fact $B(X) \sim \dfrac{b_X(L)}{L^{6g-6+2n}}$ for large L.

Recall $\exists$ Bers const $\eta = \eta(S) > 0$ s.t. each $X \in \mathcal{T}(S)$ has a pants decomp of curves w/ length $\leq \eta$.

Take $\varepsilon > 0$ small s.t. no two curves of length $\leq \varepsilon$ intersect (collar lemma). For $X \in \mathcal{T}(S)$, let

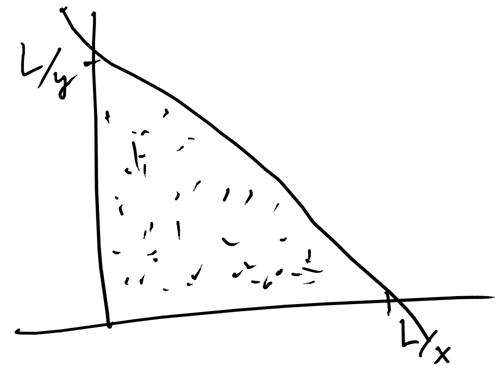
$\alpha_1, \ldots, \alpha_s$ be curves on X of len $\leq \varepsilon$, & extend to pants decomposition

$$\mathcal{P}_X = (\alpha_1, \ldots, \alpha_s, \ldots, \alpha_k), \quad k = 3g-3+n$$

s.t. $l_{\alpha_i}(X) \leq \eta$ (Bers const) $\forall i$.

Given $x, y, L > 0$, set

$$A_{x,y}(L) = \{ (m,n) \in (\mathbb{Z}_+)^2 \mid mx + ny \le L \}$$



• For $L > 6\max\{x,y\}$ have

$$|A_{x,y}(L)| \ge \frac{L^2}{12xy}$$

• For $L \ge 1$, have

$$|A_{xy}(L)| \le 3\left(\frac{L^2}{xy} + \frac{L}{\min(x,y)} + 1\right) \le 3L^2\left(\frac{1}{xy} + \frac{1}{\min(x,y)} + 1\right)$$

Now: For our fixed surf X, want to estimate

$$b_X(L) = \#\{\gamma \in \mathcal{ML}(S;\mathbb{Z}) \mid \ell_\gamma(X) \le L\}$$

using Dehn coordinate wrt pants $\mathcal{P} = \{\alpha_1, \dots, \alpha_k\}$

(Recall $\ell_{\alpha_i}(X) \le \varepsilon$ for $1 \le i \le s$
$\varepsilon < \ell_{\alpha_i}(X) \le \eta$ for $s < i \le k$)

Set $x_i = S(\ell_{\alpha_i}(X))$, $y_i = \ell_{\alpha_i}(X)$. Then

since $\ell_{\alpha_i}(X) \le \eta$ $\forall i$, get const $c = c(\eta) \ge 1$ s.t. $\forall \gamma \in \mathcal{ML}(S;\mathbb{Z})$

$$\frac{1}{c}\,\ell_\gamma(X) \le \sum_{i=1}^{k} m_i x_i + |n_i|\, y_i \le c\,\ell_\gamma(X)$$

where m_i, n_i are int# & twist parameters for γ.

Thus

each γ, $\ell_\gamma(X) \le L$ gives set $\{m_i, n_i\}$

$$\prod_{i=1}^{k} A_{x_i y_i}\left(\frac{L}{ck}\right) \le b_X(L) \le \prod_{i=1}^{k} 2A_{x_i y_i}(cL)$$

each $\{m_i, n_i\}$ with $m_i x_i + n_i y_i \le \frac{L}{ck}$ $\forall i$ & $m_i \in 2\mathbb{Z}$

$\Rightarrow$ unique curve γ with $\ell_\gamma(X) \le c\left(k\left(\frac{L}{ck}\right)\right) = L$

$$\prod_{i=1}^{k} A_{x_i y_i}\left(\frac{L}{ck}\right) \le b_X(L) \le 2\prod_{i=1}^{k} A_{x_i y_i}(cL)$$

$$\prod_{i=1}^{k} \frac{(L/ck)^2}{12\, x_i y_i} \overset{L \ge 6\max\{x_i, y_i\}}{\le} b_X(L) \overset{L\ge 1}{\le} \prod_{i=1}^{k} 2\cdot 3(cL)^2\left(1+\frac{1}{x_i y_i}+\frac{1}{\min(x_i,y_i)}\right)$$

$$C_1 \prod_{i=1}^{k} \frac{1}{x_i y_i} \le \frac{b_X(L)}{L^{6g-6+2n}} \le C_2 \prod_{i=1}^{k}\left(1+\frac{1}{x_i y_i}+\frac{1}{\min(x_i,y_i)}\right)$$

Recall $S(x) = \operatorname{arcsinh}\left(\frac{1}{\sinh(x/2)}\right)$, satisfies

$\lim_{x\to 0} \frac{S(x)}{|\log(x)|} = 1$. thus, taking $\varepsilon>0$ suff small,

may assume that for $1 \le i \le q$, $y_i = l_{\alpha_i}(X) < \varepsilon$ &

$$x_i = S(l_{\alpha_i}(X)) \sim |\log(l_{\alpha_i}(X))|$$

Thus $\frac{1}{x_i y_i} \sim \frac{1}{l_{\alpha_i}(X)\log(l_{\alpha_i}(X))} = R(l_{\alpha_i}(X))$

On the other hand, after fixing ε, may find const $D>1$ s.t.

for $s \le i \le k$, so $\varepsilon < y_i = l_{\alpha_i}(X) \le n$,

$$\frac{1}{D} \le x_i y_i = S(y_i)\, y_i \le D$$

$$\frac{1}{D} \le \min\{x_i, y_i\} \le \max\{x_i, y_i\} \le D$$

$\Rightarrow$

$$\text{Const} \prod_{\gamma : l_\gamma(X)\le\varepsilon} R(l_\gamma(X)) \underset{\text{for large } L}{\le} \frac{b_X(L)}{L^{6g-6+2n}} \underset{\text{for all } L\ge 1}{\le} \text{Const} \overbrace{\prod_{\gamma : l_\gamma(X)\le\varepsilon}\left(R(l_\gamma(X)) + \frac{1}{l_\gamma(X)}\right)}^{\text{Comparable to } \prod \frac{1}{l_\gamma(X)}}$$

Thm The function $B: \mathcal{M}(S) \to \mathbb{R}_+$ is proper and integrable over $\mathcal{M}(S)$ wrt WP-volume.

proof: Mumford Compactness thm says as $X \to \infty$ in $\mathcal{M}(S)$, (leaves compact sets)

$\inf\{ \ell_\gamma(X) \mid \gamma \text{ s.c.c on } X\} \to 0$

since $R(\varepsilon) = \dfrac{1}{\varepsilon|\log \varepsilon|} \to \infty$ as $\varepsilon \to 0$

Lower bound $B(X) \geq \prod_{\gamma : \ell_\gamma(X) \leq \varepsilon} R(\ell_\gamma(X))$

$\Rightarrow B(X) \to \infty$ as $X \to \infty$ in $\mathcal{M}(S)$

Hence B is proper.

for $X \in \mathcal{M}(S)$, let $F(X) = \prod_{\gamma : \ell_\gamma(X) \leq \varepsilon} \dfrac{1}{\ell_\gamma(X)}$.

since $B(X) \leq \text{const}\, F(X)$, to prove B integrable, suff. to prove F is integrable.

Let $\mathcal{M}^{k,\varepsilon}(S) = \{ X \in \mathcal{M}(S) \mid X \text{ has } k \text{ sccs with len} \leq \varepsilon\} \subset \mathcal{M}(S)$

May choose finitely many FN-coordinatizations $\{(\ldots, l_i, \tau_i, \ldots)\}$ of $\mathcal{T}(S)$ (one for each $\mathrm{Mod}(S)$-orbit of pants decompositions) s.t

$\mathcal{M}^{k,\varepsilon}(S)$ is covered by union of projections (to $\mathcal{M}(S)$) of the sets

$\{(\ldots, l_i, \tau_i, \ldots) \mid 0 < l_1, \ldots, l_k \leq \varepsilon,\ l_i \leq \eta,\ 0 \leq \tau_i \leq l_i\}$

Since $\displaystyle\int_{x_i=0}^{\varepsilon} \int_{y_i=0}^{x_i} \frac{1}{x_1 \cdots x_k}\, dx_1 \ldots dx_k\, dy_1 \ldots dy_k < \infty$,

follows that $\displaystyle\int_{\mathcal{M}(S)} F(X)\, d\mathrm{vol}_{WP}(X) < \infty$ ☒

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Fix top surf $S \cong S_{g,n}$, + boundary length vector $t = (t_1, \dots, t_n) \in (\mathbb{R}_{\geq 0})^n$

Teich space $\mathcal{T}_g(t)$, + moduli space $\mathcal{M}_g(t) = \mathcal{T}_g(t) / \mathrm{Mod}(S)$,

measured lam. space $\mathcal{ML}(S)$ w/ Thurston meas μ_{Th}

+ let $\mathcal{ML}(S;\mathbb{Z}) \subset \mathcal{ML}(S)$ set of integral multicurves

$X \in \mathcal{T}_g(t)$, get $B(X) = \mu_{Th}\{\lambda \in \mathcal{ML}(S) \mid \ell_\lambda(X) \leq 1\}$

$L > 0$ $\qquad b_X(L) = \#\{\gamma \in \mathcal{ML}(S;\mathbb{Z}) \mid \ell_\gamma(X) \leq L\}$

$$\frac{b_X(L)}{L^{6g-6+2n}} \to B(X) \quad \text{as } L \to \infty$$

Saw

$$\frac{b_X(L)}{L^{6g-6+2n}} \leq \prod_{\substack{\gamma : scc \\ \ell_\gamma(X) \leq \varepsilon}} \frac{1}{\ell_\gamma(X)} = \underbrace{F(X)}_{\text{Integrable!}}$$

Set $f_L(X) = \dfrac{b_X(L)}{L^{6g-6+2n}}$. Thus

<u>Cor</u> Seq $\{f_L\}_L$ functions on $\mathcal{T}_g(t)$ satisfies Lebesgue dominated convergence (uniformly bdd above by integrable function $F(X)$). Hence pointwise limit $B(X) = \lim_{L \to \infty} f_L(X)$ is integrable +

$$\int f_L(X)\,dX \to \int B(X)\,dX.$$

For $\gamma \in \mathcal{ML}(S;\mathbb{Z})$, set $s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \mid \ell_\alpha(X) \leq L\}$

Then $S_X(L,\gamma) \le b_X(L)$, so $\dfrac{S_X(L,\gamma)}{L^{6g-6+2n}}$ also satisfies Lebesgue dom. conv.

<u>Def</u> $$P(L,\gamma,t) = \int_{\mathcal{M}_g(t)} S_X(L,\gamma)\, dX$$

<u>Thm</u> for any $\gamma = \sum_{i=1}^{k} c_i \gamma_i \in \mathcal{ML}(S,\mathbb{Z})$, integral of $S_X(L,\gamma)$ over $\mathcal{M}_g(t)$ is

$$P(L,\gamma,t) = \frac{1}{|Sym(\gamma)|} \int_{\gamma\cdot x \le L} Vol_{wp}\Big(\overbrace{\mathcal{M}(S\setminus\gamma;\ \ell_\gamma = x,\ \ell_\beta = b}^{\text{moduli space of cut open surf w specified bdry lengths}}\Big)\, x\, dx$$

where $x = (x_1,\dots,x_k)$ & $\gamma\cdot x = \sum c_i x_i$

<u>proof</u>: Let $f:\mathbb{R}\to\mathbb{R}$ be the function $\chi_{[0,L]}$. so

$$f(z) = \begin{cases} 1, & 0 \le z \le L \\ 0 & z > L \end{cases}$$

get function $f_\gamma : \mathcal{T}_g(t) \to \mathbb{R}$
$$f_\gamma(X) = \sum_{h\in Mod/Stab(\gamma)} f(\ell_{h\cdot\gamma}(X)),$$ descends to $f_\gamma : \mathcal{M}_g(t)\to\mathbb{R}$

Covolume formula:

$$\int_{\mathcal{M}_g(t)} f_\gamma = \frac{1}{|Sym(\gamma)|}\int_{x\in\mathbb{R}_+^k} f(|x|)\, Vol(S\setminus\gamma;\ \ell_\gamma = x,\ \ell_\beta = t)\, x\, dx$$

$$= \frac{1}{Sym(\gamma)} \int_{\gamma\cdot x\le L} Vol(S\setminus\gamma;\ \ell_\gamma = x,\ \ell_\beta = t)\, x\, dx$$

But rather for $X \in \mathcal{T}_g(t)$

$$S_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \mid \ell_\alpha(X) \le L\}$$

$$= \sum_{\{h \in \mathrm{Mod}(S)/\mathrm{Stab}(\gamma) \,:\, \ell_{h\cdot\gamma}(X) \le L\}} 1$$

$$= \sum_{h \in \mathrm{Mod}(S)/\mathrm{Stab}(\gamma)} f(\ell_{h\cdot\gamma}(X)) = f_\gamma(X).$$

Thus $\int_{\mathcal{M}_g(t)} f_\gamma(X) = \int_{\mathcal{M}_g(t)} S_X(L,\gamma)$ $\boxtimes$

Cor for any $\gamma \in \mathcal{ML}(S;\mathbb{Z})$, $P(L,\gamma,t)$ is a polynomial of degree $6g-6+2n$ in L and t. coeff of $L^{6g-6+2n}$ is

set $$C_\gamma = \lim_{L\to\infty} \frac{P(L,\gamma,t)}{L^{6g-6+2n}} > 0$$

this is a positive rational # independent of t. (just depends on γ).

Def fixing g, n, $t = (t_1,\dots,t_n)$, set
$$b_{g,n}(t) = \int_{\mathcal{M}_g(t)} B(X)$$

Thm $b_{g,n}(t)$ is a positive number in $\mathbb{Q}\cdot\pi^{6g-6+2n}$, independent of t.
(skip proof). write just $b_{g,n}$

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Finally we may count asymptotic growth of # simple closed curves.

In fact, may count in each orbit separately.

<u>Main Thm</u> (Mirzakhani) for any multicurve $\gamma \in \mathcal{ML}(S;\mathbb{Z})$, & $X \in \mathcal{T}_g(t)$, we have

$$\boxed{S_X(L,\gamma) \sim \frac{B(X)}{b_{g,n}} c_\gamma L^{6g-6+2n} \quad \text{as } L \to \infty}$$

<u>Rmk</u>: $b_{g,n}$ universal. $B(X)$ only depends on X (not L or γ), c_γ only depends on γ (not X).

<u>Cor</u> for any pair $\gamma_1, \gamma_2 \in \mathcal{ML}(S;\mathbb{Z})$

$$\frac{S_X(L,\gamma_1)}{S_X(L,\gamma_2)} \to \frac{c_{\gamma_1}}{c_{\gamma_2}}$$

<u>Cor</u> If $S_X(L)$ = # simple closed curves on X with $\ell_\gamma(X) \leq L$.

Then
$$\lim_{L\to\infty} \frac{S_X(L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \left(c_{\gamma_1} + \cdots + c_{\gamma_k} \right)$$

where $\gamma_1, \dots, \gamma_k$ are representatives for the finitely many $\mathrm{Mod}(S)$-orbits of SCCs in S

<u>Proof</u>: up to action of $\mathrm{Mod}(S)$, there are only finitely many SCCs on S. If $\gamma_1, \dots, \gamma_k$ are orbit representatives, then

$$\#\{\gamma \text{ scc on } S \mid \ell_\gamma(X) \leq L\} = S_X(L,\gamma_1) + \cdots + S_X(L,\gamma_k) \quad \blacksquare$$

Discrete Measures Measured Lamination Space:

given $\gamma \in \mathcal{ML}(S;\mathbb{Z})$ & $L>0$, def meas on $\mathcal{ML}(S)$:

$$\nu_L^\gamma = \frac{1}{L^{6g-6+2n}} \sum_{\substack{\alpha \in \mathrm{Mod}(S)\cdot\gamma \\ \subset \mathcal{ML}(S)}} \delta_{\frac{\alpha}{L}}$$

(Sum of dirac measures at $\frac{\alpha}{L} \in \mathcal{ML}(S)$.)

Measure Convergence Thm

for any $\gamma \in \mathcal{ML}(S;\mathbb{Z})$, the measures ν_L^γ weak * converge to $\frac{c_\gamma}{b_{g,n}} \cdot \mu_{Th}$ — Thurston measure. That is

$$\int f \, \nu_L^\gamma \to \int f \, \mu_{Th} \quad \text{for any } f \in \underbrace{C_c(\mathcal{ML}(S))}_{\text{Cont w/ compact support}}$$

Proof of main Thm assuming measure convergence:

$$S_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \mid \ell_\alpha(X) \le L\}$$

$$= \#\{\underbrace{(\mathrm{Mod}(S)\cdot\gamma) \cap (L\cdot B_X)}_{\text{subset } \mathcal{ML}(S)}\} \qquad B_X = \{\lambda \in \mathcal{ML} \mid \ell_\lambda(X) \le 1\}$$

$$= \Big(\sum_{\alpha \in \mathrm{Mod}(S)\cdot\gamma} \delta_\alpha\Big)(L\cdot B_X)$$

$$= \Big(\sum_{\alpha \in \mathrm{Mod}(S)\cdot\gamma} \delta_{\frac{\alpha}{L}}\Big)(B_X) = L^{6g-6+2n}\,\nu_L^\gamma(B_X)$$

so

$$\frac{S_X(L,\gamma)}{L^{6g-6+2n}} = \nu_L^\gamma(B_X) \to \frac{c_\gamma}{b_{g,n}}\,\mu_{Th}(B_X) = \frac{c_\gamma}{b_{g,n}}\,B(X) \qquad \blacksquare$$

(∂B_X has measure 0: approx χ_{B_X} with cont. functions.)

[7-16] — End 4/8

$\boxed{\text{Mon 4/23/18}}$

Remarks to prove measure convergence Theorem.

<u>Lemma</u> The seq $\{\nu_L^\gamma\}$ is weakly normal: any subsequence contains a weakly convergent subsequence.

<u>Proof</u>: Recall space of Prob. measures on any compact space is compact
$\Rightarrow$ any seq has a weak-$*$ convergent subsequence.

So, to prove lemma, suffices to show:

$$\forall K \subset \mathcal{M}\mathcal{L}(S) \text{ compact}, \quad \sup_L \nu_L^\gamma(K) < \infty$$

Fix K compact. WLOG, $K = T \cdot B_{x_0}$, some $x_0 \in \mathcal{T}(S)$.

Then

$$\nu_L^\gamma(K) = \nu_L^\gamma(T B_{x_0}) = \frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{Mod}\cdot\gamma} \delta_{\frac{\alpha}{L}}(T \cdot B_{x_0})$$

$$= \frac{1}{L^{6g-6+2n}} \# \left\{ \alpha \in \mathrm{Mod}\cdot\gamma \;\middle|\; \ell_{x_0}\left(\frac{\alpha}{L}\right) \leq T \right\}$$

$$\leq \frac{1}{L^{6g-6+2n}} \# \left\{ \alpha \text{ scc} \;\middle|\; \ell_{x_0}(\alpha) \leq T \cdot L \right\}$$

$$= \frac{b_{x_0}(T \cdot L)}{L^{6g-6+2n}}$$

Recall: $b_x(L) = \{ \alpha \text{ scc} \mid \ell_\alpha(x) \leq L \}$

$$= T^{6g-6+2n} \left(\frac{b_{x_0}(TL)}{(TL)^{6g-6+2n}} \right)$$

$$\leq T^{6g-6+2n} \prod_{\substack{\gamma \text{ scc} \\ \ell_\gamma(x_0) \leq \varepsilon}} \frac{1}{\ell_\gamma(x_0)}$$

function $< \infty$, indep of L. $\blacksquare$

Lem Any weak subsequential limit of $\{\nu_L^\gamma\}$ is invariant under $\mathrm{Mod}(S)$ on $\mathcal{ML}(S)$ action and absolutely cont wrt μ_{Th}.

Proof:

Assume $L_i \to \infty$ subseq st $\nu_{L_i}^\gamma \to \eta$ in weak * top.

$\mathrm{Mod}(S)$-invariance of η is clear: for $\phi \in \mathrm{Mod}(S)$,

$$\phi_*(\nu_L^\gamma) = \phi_*\left(\frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{Mod}(S)\cdot\gamma} \delta_{\frac{\alpha}{L}}\right)$$

$$= \frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{Mod}\cdot\gamma} \delta_{\frac{\phi(\alpha)}{L}} = \frac{1}{L^{6g-6+2n}} \sum_{\beta \in \mathrm{Mod}\cdot\gamma} \delta_{\beta/L} = \nu_L^\gamma$$

thus $\nu_{L_i}^\gamma \to \eta \Rightarrow$

$$\nu_{L_i}^\gamma = \phi_*(\nu_{L_i}^\gamma) \to \phi_*(\eta),$$ so must have $\phi_*(\eta) = \eta$.

To prove absolute continuity:

Let $U \subset \mathcal{ML}(S)$ be a convex open set in some train track chart U_τ, $\tau \subset S$ a max, recurrent train track.

Then we've seen that for

$$b_\tau(U,L) = \#(\mathcal{ML}(S;\mathbb{Z}) \cap (L\cdot U) \cap U_\tau)$$

we have $$\frac{b_\tau(U,L)}{L^{6g-6+2n}} \to \mu_{Th}(U).$$ Thus

$$\eta(U) \le \liminf_{i\to\infty} \nu_{L_i}^\gamma(U) \le \liminf_{i\to\infty} \frac{b_\tau(U,L_i)}{L_i^{6g-6+2n}} = \mu_{Th}(U).$$

Now, if $A \subset \mathcal{ML}(S)$ any set with $\mu_{Th}(A)=0$, we can approx A by finite union of open sets U as above. Hence $\eta(A) \le \mu_{Th}(A) = 0$.

Thus η is absolutely cont to μ_{Th}. ☒

7-18

Fact that μ_{Th} is ergodic for the action $\mathrm{Mod}(S) \curvearrowright \mathcal{ML}(S)$

$\Rightarrow$ every $\mathrm{Mod}(S)$-invt meas η abs. cont wrt μ_{Th}

has form $\eta = c \cdot \mu_{Th}$ for some $c \geq 0$.

abs cont $\Rightarrow \exists f \in L^1(\mathcal{ML}, \mu_{Th})$ st $\eta(A) = \int_A f \, d\mu_{Th}$ $\forall$ meas $A \subset \mathcal{ML}$

(f is Radon-Nikodym derivative)

η Mod-invt $\Rightarrow$ f is Mod-invt a.e.

Hence μ_{Th} ergodicity $\Rightarrow$ f is a.e. constant, so $\eta = \text{const}\, \mu_{Th}$

Now, let $L_i \to \infty$ be any subsequence st $\nu_{L_i}^{\gamma}$ weakly converges to some meas η on $\mathcal{ML}$. By above, may write $\eta = h \cdot \mu_{Th}$ some $h \geq 0$.

Suffices to show: $\boxed{h = \dfrac{c_\gamma}{b_{g,n}}}$ independent of the subsequence $L_i \to \infty$.

$t = (t_1, \dots, t_n)$ boundary lengths

Proof: for any $X \in \mathcal{T}_{g,n}(S, t)$ $B_X = \{\lambda \in \mathcal{ML} \mid \ell_\lambda(X) \leq 1\}$

$$\nu_{L_i}^{\gamma}(B_X) = \frac{\#\{\alpha \in \mathrm{Mod}\cdot\gamma \mid \ell_\alpha(X) \leq L\}}{L_i^{6g-6+2n}} = \frac{s_X(\gamma, L_i)}{L_i^{6g-6+2n}}$$

since $\mu_{Th}(\partial B_X) = 0$, we can approx char function χ_{B_X} by continuous functions to get

$$\frac{s_X(\gamma, L_i)}{L_i^{6g-6+2n}} = \nu_{L_i}^{\gamma}(B_X) \to h \cdot \mu_{Th}(B_X) = h \cdot B(X).$$

Recall: as function of $X \in \mathcal{T}_{g,n}(S, t)$

$$\frac{s_X(\gamma, L)}{L^{6g-6+2n}} \leq f_L(X) = \frac{b_X(L)}{L^{6g-6+2n}} \leq F(X) = \prod_{\substack{\gamma \text{ scc} \\ \ell_\gamma(X) < \varepsilon}} \frac{1}{\ell_\gamma(X)}$$

and that $F(X)$ is integrable.

Hence: by Lebesgue Dominated Convergence:

$$h \cdot b_{g,n} = h \int_{\mathcal{M}(S)} B(X)\, dVol_{wp}(X) = \int_{\mathcal{M}(S)} \lim_{i\to\infty} \frac{S_X(\gamma, L_i)}{L_i^{6g-6+2n}}\, dVol_{wp}(X)$$

$$= \lim_{i\to\infty} \int_{\mathcal{M}(S)} \frac{S_X(\gamma, L_i)}{L_i^{6g-6+2n}}\, dVol_{wp}(X)$$

$$= \lim_{i\to\infty} \frac{P(L_i, \gamma, t)}{L_i^{6g-6+2n}} = c_\gamma$$

by defn c_γ = coeff of $L^{6g-6+2n}$ in poly $P(L, \gamma, t)$.

hence const $h = \dfrac{c_\gamma}{b_{g,n}}$ is indep of subseq.

Thus any conv subseq $\{\nu_{L_i}^\gamma\}_{L_i\to\infty}$ converges to $\dfrac{c_\gamma}{b_{g,n}} \mu_{Th}$.

So conclude $\{\nu_L^\gamma\}_{L>0}$ weak-* converges to $\dfrac{c_\gamma}{b_{g,n}} \mu_{Th}$

This proves measure convergence Thm & finishes proof of main counting theorem! $\boxtimes$