VI - Measured Laminations

Our next (and probably final) goal is to calculate the asymptotic growth rate of the number of simple closed curves on a fixed hyp surf. For this, we need to first develop the theory of measured laminations.

Fix a hyperbolic surface X. maybe for now assume no boundary.

so $X = \mathbb{H}^2/\Gamma$ $\quad \Gamma \leq PSL(2,\mathbb{R})$ acting discretely & freely.

& $\tilde{X} = \mathbb{H}^2$, $\quad \mathbb{H}^2 \to X$

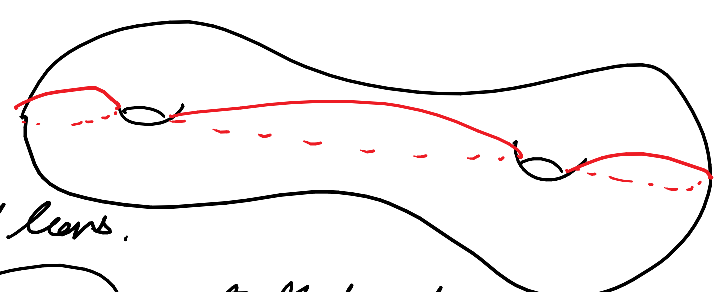
<u>Def</u> By a <u>geodesic</u> on X we mean image of a complete geodesic in $\mathbb{H}^2$

geodesic <u>simple</u> if it has no self intersections.

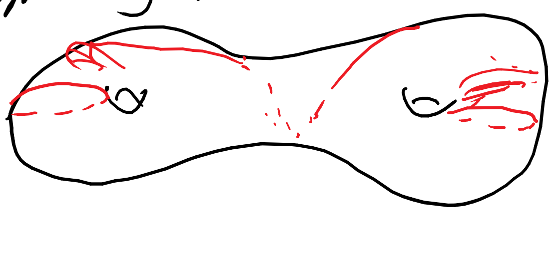
<u>Def</u> a <u>geodesic lamination</u> on X is a closed subset $L \subset X$ that is a disjoint union of geodesics.

- geodesics comprising L are its <u>leaves</u>

<u>Ex</u> union of finitely many disjoint simple closed geodesics is a lamination:



. can include infinite leaves spiralling towards closed leaves.



☆ Most geod. laminations do <u>not</u> look like this.

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Lem The closure of a non-empty disjoint union L of geodesics is a lamination.

Proof: Show $\bar{L}$ = disjoint union of geodesics

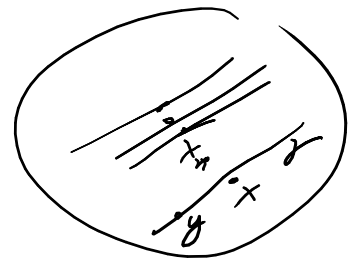
say $x \in \bar{L}$, choose $x_n \in L$, $x_n \to x$

each $x_n \in L$, so x_n lies on a geodesic γ_n of L

$\rightsquigarrow$ picks out direction $d_n \in \mathbb{P}T_{x_n}X$

$\mathbb{P}T_xX$ cpct, so pass to subseq s.t.

$d_n \to$ direction $d \in \mathbb{P}T_xX$



Let γ = geodesic through x in direction d.

Claim: $\gamma \subset \bar{L}$:

Let $y \in \gamma$ be signed dist h from x

Let y_n = pt on γ_n signed dist h from x_n

then $x_n \to x$ & $d_n \to d$ $\Rightarrow$ $y_n \to y$. hence $y \in \bar{L}$ ✓

Thus $\bar{L}$ is a union of geodesics. Remains to show

$\bar{L}$ disjoint union of simple geodesics:

Say 2 (possibly equal) geods $\gamma, \gamma' \subset \bar{L}$ intersect transversally at x. Find geods $\beta, \beta' \subset L$ passing close to x & approximating directions of γ & γ' arbitrarily well

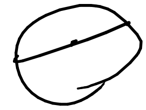
$\Rightarrow$ can find such β, β' that intersect, contradiction. ∎

6-2

Def A leaf γ of a lamination L is *isolated*, if

$\forall x \in \gamma$, $\exists$ nbhd U of x s.t.

$(U, U \cap L) \cong (\text{disk}, \text{diameter})$



Exercise: holds for every $x \in \gamma$ $\Leftrightarrow$ holds for some $x \in \gamma$.

|| set of x where it holds is open & closed in γ. ||

The *derived lamination* of L is

$$L' = L - \text{isolated leaves} \quad (\text{provided nonempty})$$

- L' is closed, thus L' is a geod lamination (provided nonempty)

Lemma If L' empty, then L is a finite union of simple closed geodesics

Pf: $L' = \emptyset \Rightarrow$ each leaf isolated $\Rightarrow$ L closed 1-submfld X, hence disjoint union of simple closed geods. ⊠

Recall: If (X, d), the Hausdorff distance between subsets $A, B \subset X$ is:

$$d_{Haus}(A,B) = \inf\{\varepsilon > 0 \mid A \subset Nbhd_\varepsilon(B) \ \& \ B \subset Nbhd_\varepsilon(A)\}$$

- defines a metric on the set $C(X)$ of compact subsets
- If X compact, so is $(C(X), d_{Haus})$

Rmk If X = our fixed hyp surf. Then

- X cpct (i.e., no punctures) $\Rightarrow$ every lamination $L \subset X$ cpct.
- If X not compact, then a lamination L is compact provided it doesn't have leaves going straight out a cusp.

Def Let $\mathcal{GL}(X)$ = set of geodesic laminations on X

Prop If X closed hyp surf (compact, no boundary) then $\mathcal{GL}(X)$ is a closed subset of $(\mathcal{C}(X), d_{Haus})$. Hence $\mathcal{GL}(X)$ is compact in top from Hausdorff distance.

- If X has punctures, similar result holds if you restric to compact geodesic laminations.

pf: omit, somewhat technical, but you can imagine how it must go.
we do not need it.

Alternate defn of geodesic lamination:

$X = \mathbb{H}^2/\Gamma$, $\tilde{X} = \mathbb{H}^2$. write $\partial\mathbb{H}^2 = \hat{\mathbb{R}} \cong S^1$ for boundary at ∞.

If $L \subset X$ is a geodesic lamination, then preimage $\tilde{L} \subset \mathbb{H}^2$ is a closed, disjoint union of complete (bi-infinite) geodesics in $\mathbb{H}^2$. Hence: a geodesic lamination of $\mathbb{H}^2$.

Further: $\tilde{L}$ is invariant under action $\Gamma \cong \pi_1(X)$.

Note: a complete geod in $\mathbb{H}^2$ equiv to unordered pair of distinct pts in $\partial\mathbb{H}^2$. Thus:

"double boundary"

set of complete geodesics in $\mathbb{H}^2$ $\Longleftrightarrow$ $\partial^2\mathbb{H}^2 :=$ set of 2-elt subsets of $\partial\mathbb{H}^2$

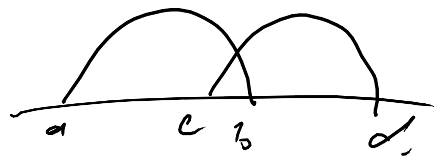
$= (\partial\mathbb{H}^2) \times (\partial\mathbb{H}^2) \setminus \text{diagonal} \big/ (a,b) \sim (b,a)$

$\partial^2\mathbb{H}^2$ has natural topology

Since $\tilde{L}$ is a union of geodesics, we may think of $\tilde{L} \subset \partial^2 \mathbb{H}^2$.
— write as $\partial \tilde{L} \subset \partial^2 \mathbb{H}^2$

What properties does it have?

- Say 2 pts $\{a,b\}, \{c,d\} \in \partial^2 \mathbb{H}^2$ are <u>linked</u> if their geodesics in $\mathbb{H}^2$ intersect transversely



<u>Thus:</u> every pair of pts in $\partial \tilde{L} \subset \partial^2 \mathbb{H}^2$ are unlinked.

<u>Lemma</u> Let $\Lambda \subset \mathbb{H}^2$ be a union of complete geodesics in $\mathbb{H}^2$ (so, think $\Lambda \subset \partial^2 \mathbb{H}^2$).

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Then Λ closed in $\mathbb{H}^2$ iff $\partial \Lambda$ closed in $\partial^2 \mathbb{H}^2$.

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proof: $\Leftarrow$ Assume $\partial\Lambda$ closed. Say $p_i \in \Lambda$ accumulate to $p \in \mathbb{H}^2$.

Choose geodesic $\gamma_i \subset \Lambda$ with $p_i \in \gamma_i$

$\partial \gamma_i = \{x_i, y_i\}$

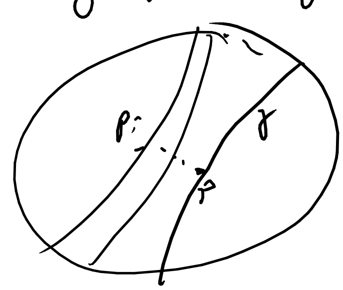
compactness $S^1 \times S^1 \Rightarrow$ may pass to sub ordered pairs (x_i, y_i) converge to $(x,y) \in S^1 \times S^1$.

since $p_i \to p$, $p_i \in \gamma_i$, must have $x \neq y$.

$\partial\Lambda$ closed $\Rightarrow \{x,y\} \in \partial\Lambda \Rightarrow$ geod γ fm x to y in Λ

now clear that $p \in \gamma$

(else, have nbhd of p missing γ & all but finitely many of the γ_i)



$\Rightarrow$ Assume Λ closed. Say $\partial\gamma_i = \{x_i, y_i\} \in \partial\Lambda \subset \partial^2\mathbb{H}^2$

converges to $\{x, y\} \in \partial^2\mathbb{H}^2$.

Fix parametrization $\gamma: \mathbb{R} \to \mathbb{H}^2$ of geod from x to y.

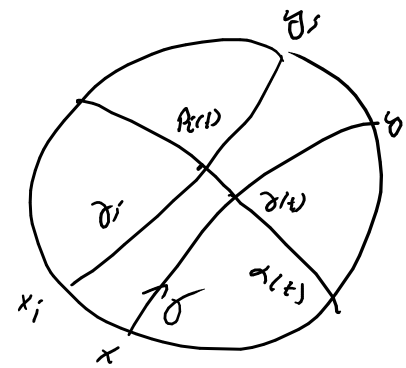
For each $t \in \mathbb{R}$ let α_t = geod perp to γ at $\gamma(t)$

Since α_t separates x from y &

$\partial\gamma_i = \{x_i, y_i\} \to \{x, y\}$,

for all large i, γ_i & α_t must intersect

transversally at some pt $p_i(t) \in \Lambda$



notice that $\{x_i, y_i\} \to \{x, y\}$

$\Rightarrow$ $p_i(t)$ converges to $p(t) = \alpha_t \cap \gamma. = \gamma(t)$

Λ closed $\Rightarrow$ $\gamma(t) = p(t) \in \Lambda$

Holds for all $t \Rightarrow \gamma \subset \Lambda$,

Thus $\{x, y\} = \partial\gamma \in \partial\Lambda$, showing $\partial\Lambda$ closed $\boxtimes$

Finally: notice $\pi_1(X) \cong \Gamma \leq PSL(2,\mathbb{R})$ acts on $\partial^2\mathbb{H}^2$ (diagonally).

If $\Lambda \subset \mathbb{H}^2$ union of complete geodesics then

$\Lambda \subset \mathbb{H}^2$ Γ-invariant $\Leftrightarrow$ $\partial\Lambda \subset \partial^2\mathbb{H}^2$ Γ-invariant.

Thus we have:

Thm Natural bijections:

L geodesic lamination on X	$\longleftrightarrow$	$\tilde{L} \subset \mathbb{H}^2$ closed, Γ-invt union of mutually disjoint complete geodesics	$\longleftrightarrow$	$\partial\tilde{L}$ closed, Γ-invt subset of $\partial^2\mathbb{H}^2$

$\boxtimes$

6-6

Topology of laminations:

Let $\lambda \subset X$ be geod lamination.

open subset $X \setminus \lambda$ with induced path metric

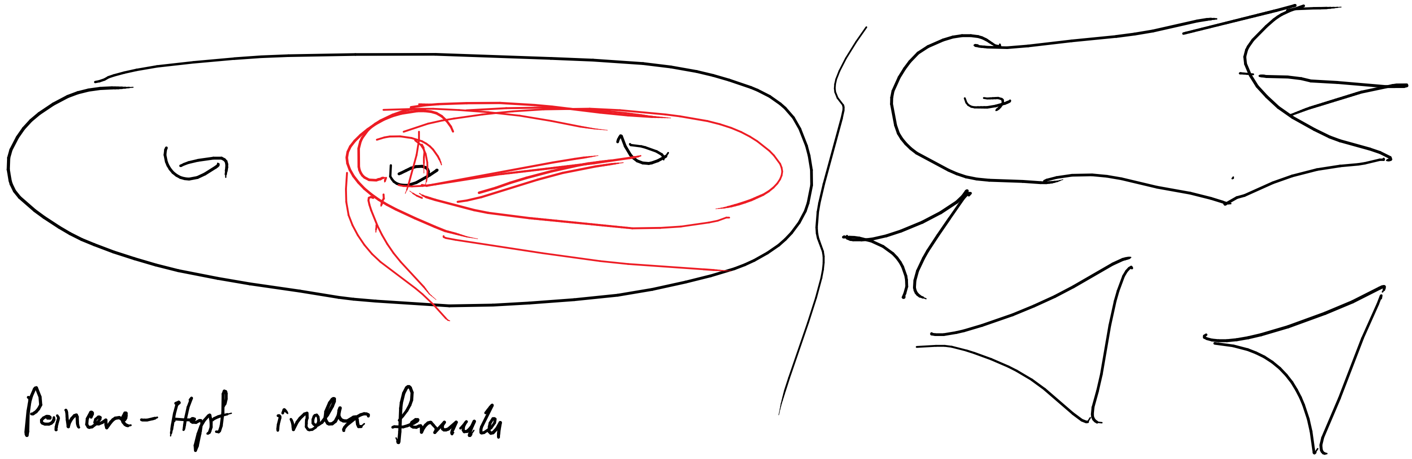
$\overline{X \setminus \lambda}$ metric completion:

↳ hyperbolic surface with geodesic boundary.

X has finite area $(-2\pi \chi(X))$, so

$\overline{X \setminus \lambda}$ has finite area + finitely many components.

- each component potentially has:
  - compact part
  - cusps of X
  - finitely many "spikes"



Poincare-Hopf index formula

$$\chi(\overline{X \setminus \lambda}) = \chi(X) + \tfrac{1}{2} \# \text{spikes}$$

$\Rightarrow$ # of components of $\overline{X \setminus \lambda}$ unif. bdd,

$\Rightarrow \lambda$ can have only finitely many minimal sublaminations

Def a lamination λ is minimal if it does not contain any proper sublaminations.

$\Leftrightarrow$ every half-leaf in λ is dense.

Structural Prop: A geodesic lamination is the union of finitely many minimal sublaminations and of finitely many infinite isolated leaves, whose ends spiral along a minimal sublamination or converge to a cusp.

Def Let L be a geod. lamination on X.

Let $\tau(L)$ = set of transversals to L

= compact 1-manifolds embedded in X, that are transverse to L + with boundary (if any) in $X \setminus L$

(Birman-Series) union of all simple geodesics on X has Hausdorff dim 1

$\Longrightarrow$ Each geodesic lamination L has Hausd dim 1

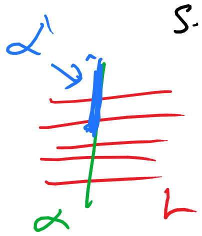
Cor If L has no isolated leaves, then $\forall$ transversal $\alpha \in \tau(L)$

$\alpha \cap L \cong$ Cantor set. (closed, Haus-dim 0 totally disconnected, no isolated pts)

A transverse measure on the geodesic lamination L is

is an assignment of a Radon measure on each arc $\alpha \in \tau(L)$

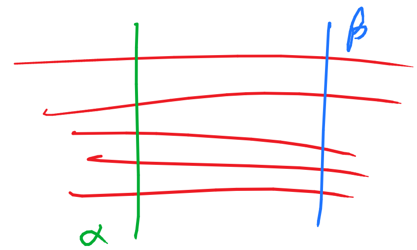
↳ finite non-neg mass to each rel cpct Borel, countably additive?



s.t.: · If $\alpha' \subset \alpha$ subtransversal, then

measure of α' = restriction of measure of α.

· If transversals $\alpha, \beta \in \tau(L)$ are homotopic through transversals, then the homotopy sends one measure to the other



Note: Immediately implies that for any transversal $\alpha \in \tau(L)$, support of measure contained in $\alpha \cap L$.

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Ex $L = \gamma_1 \cup \cdots \cup \gamma_n$ finite union of SCCS,

then transverse measure on L equiv to

choice of real #'s $c_1, \dots, c_n \geq 0$:

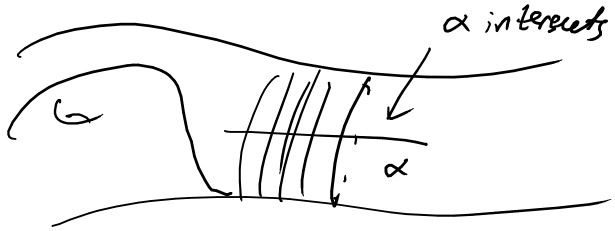
For any arc $\alpha \in \mathcal{T}(L)$, put measure

$$\mu(A) = \sum_{i=1}^{n} c_i \, \#(A \cap \gamma_i) \quad \text{for any subset } A \subset \alpha.$$

cardinality.

Fact: An infinite isolated leaf of L cannot be in the support of any transverse measure on L

pf: such a leaf would necessarily provide infinite mass to certain compact transversals $\alpha \in \mathcal{T}(L)$



α intersects leaf ∞-many times, & each time contributes some positive measure!

Prop Every geod lamination L admits a transverse measure whose support consists of all the minimal sublaminations of L.

A measured geodesic lamination is a compact geodesic lamination L equipped with a transverse measure whose support is all of L.

notation: . μ for measured lamination & $\lambda_\mu = \operatorname{supp}(\mu)$ the support underlying geodesic lamination.

. $\mathcal{ML}(X)$ = set of all measured laminations on X (will give topology soon!)

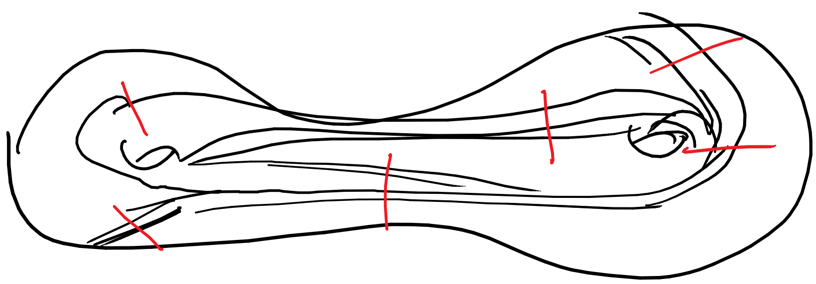
The length of a measured lamination:

Let μ be a measured lamination on X with support λ_μ.

$l_\mu(X)$ = length of μ on $X \in \mathbb{R}_+$ is defined as:

Pick finite family of transversals $\alpha_1, \dots, \alpha_n \in \tau(\lambda_\mu)$

s.t. $\lambda_\mu - \bigcup_i \alpha_i$ consists of geodesic arcs finite length



each component of $\lambda_\mu - \bigcup_i \alpha_i$ is determined by endpts in $\bigcup_i \alpha_i$

$\Rightarrow$ measure that μ assigns to $\bigcup_i \alpha_i$ induces

measure η on set of components of $\lambda_\mu - \bigcup_i \alpha_i$

(htpy invariance $\Rightarrow$ doesn't matter which endpts we choose)

length $\boxed{\displaystyle l_\mu(X) := \int_{\substack{l \text{ arc of} \\ \lambda_\mu - \bigcup_i \alpha_i}} \text{length}(l)\, d\eta(l)}$

invariance of transversal measures under htpy + subdivision $\Rightarrow$ $l_\mu(X)$ well defined, indep of $\bigcup_i \alpha_i$.

Ex If λ_μ consists of isolated leaves ($\lambda_\mu' = \emptyset$),

so λ_μ = finite union of SCCs $\gamma_1 \cup \dots \cup \gamma_n$, &

μ = choices pos # $c_i > 0$ each i, then

$$l_\mu(X) = \sum_{i=1}^n c_i l_{\gamma_i}(X)$$

generalization of length of SCC!

6-10

Alternate defn of measured lamination:

geodesic lamination L on $X = \mathbb{H}^2/\Gamma$ $\Leftrightarrow$ closed, unlinked, Γ-invnt subset $\partial^2(\mathbb{H}^2)$.

Prop A measured lamination on X is equiv. to a

a Γ-invnt Radon measure μ on $\partial^2(\mathbb{H}^2)$
s.t. $\mathrm{supp}(\mu) \subset \partial^2(\mathbb{H}^2)$ is unlinked (i.e., a lamination)

(skip proof, one dir $\Leftarrow$ sort of clear)

Independence on the hyperbolic structure.

To make all these definitions of geodesic laminations + measured laminations we have specified a hyperbolic structure. But in fact:

Prop Let S be a surf + $X, X' \in \mathcal{T}(S)$ two hyperbolic structures on S, with markings

$$S \xrightarrow{f} X, \qquad S \xrightarrow{f'} X'.$$

Then the change of markings

$f' \circ f^{-1} : X \to X'$ induces canonical bijections

$$\mathcal{GL}(X) \leftrightarrow \mathcal{GL}(X') \;+$$

$$\mathcal{ML}(X) \leftrightarrow \mathcal{ML}(X')$$

+ furthermore a canonical homeo

$\partial \tilde{X} \to \partial \tilde{X}'$, equivariant wrt action of $\pi_1(S)$

Proof idea: may adjust $f' \circ f^{-1}: X \to X'$ by a htpy to get a bilipschitz map $h: X \to X'$ wrt the 2 metrics.

Key pt: $\Rightarrow$ lift $\tilde{h}: \tilde{X} \to \tilde{X'}$ is $\pi_1(S) \cong \pi_1(X) \cong \pi_1(X')$ equivariant quasi-isometry $\tilde{h}: \mathbb{H}^2 \to \mathbb{H}^2$

(i.e, $\exists K \geq 1, C \geq 0$ s.t, $\forall a, b \in \mathbb{H}^2$

$\frac{1}{K} d(a,b) - C \leq d(\tilde{h}(a), \tilde{h}(b)) \leq K d(a,b) + C$)

$\Rightarrow$ for every complete geodesic γ in $\mathbb{H}^2$, $\tilde{h}(\gamma)$ is a (K,C)-quasi-geodesic, hence stays within unifm dist $D = D(K,C)$ of a unique comple geodesic γ' on $\mathbb{H}^2$.

upshot: • $\tilde{h}$ extends to an equivariant homeo

$$\begin{array}{ccc} \partial\tilde{h}: \partial\mathbb{H}^2 & \to & \partial\mathbb{H}^2 \\ \parallel & & \parallel \\ \partial\tilde{X} & & \partial\tilde{X}' \end{array}$$

• for every complete geodesic α on X, there is a unique geodesic α' on X' st $h(\alpha)$ may be deformed to α' via a homotopy that moves pts a uniformly bounded distance.

Further, this correspondence $\alpha \longleftrightarrow \alpha'$ preserves simpleness, disjointness, etc. $\boxtimes$

Cor for S orq surf with $\chi(S) < \infty$ (so S admits hyp metric) we may unambiguously define objects $\mathcal{GL}(S)$ & $\mathcal{ML}(S)$.

⟵ End 3/21/18

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Get embedding:

$$\{\text{simple closed curves on } S\} \longrightarrow \mathcal{ML}(S)$$

$\gamma \longmapsto$ γ thought of as geodesic lam, equipped with transverse measure $1 \cdot \gamma$

(i.e, measure on $\alpha \in \tau(\gamma)$ is

$$\sum_{p \in \alpha \cap \gamma} \delta_p$$ — dirac measure)

Length function: we now see that

every measured lamination $\mu \in \mathcal{ML}(S)$ has a length function $l_\mu : \mathcal{T}(S) \to \mathbb{R}_+$ def by:

for $X \in \mathcal{T}(S)$, realize μ as $\tilde{\mu} \in \mathcal{ML}(X)$

& set $l_\mu(X) =$ length of geodesic measured lamination $\tilde{\mu}$ on X.

Intersection Pairing

for γ a scc on S & $\mu \in \mathcal{ML}(S)$, def intersection #

$$i(\gamma,\mu) = \int_{\gamma'} d\mu_{\gamma'}, \quad \text{where } \gamma \sim \gamma' \in \tau(\lambda_\mu) \text{ \&}$$

$\mu_{\gamma'} :=$ measure μ assigns to γ'

— generalize usual "geometric intersection number" of simple closed curves

Topology on $\mathcal{ML}(S)$: minimal top s.t. $i(\gamma,\cdot)$ continuous for each scc γ on S.

$\mathbb{R}_+ \curvearrowright \mathcal{ML}(S)$ by scaling the transverse measure.

<u>Def</u> Projective measured lamination space

$$\mathcal{PML}(S) = \mathcal{ML}(S) / \mathbb{R}_+$$

<u>Facts:</u>

1) length function $l_\mu : \mathcal{T}(S) \to \mathbb{R}_+$ is continuous $\forall \mu \in \mathcal{ML}(S)$

2) int #'s give map

$$\mathcal{ML}(S) \longrightarrow (\mathbb{R}_+)^{\mathcal{S}(S)}$$
$$\mu \longmapsto (i(\gamma,\mu))_{\gamma \in \mathcal{S}(S)}$$

— Set of scc's

⊛ injective. Hence an embedding.

descends to $\quad \mathcal{PML}(S) \to \mathbb{PR}_+^{\mathcal{S}(S)}$

3) (Thurston) the composition

$$\mathcal{T}(S) \to \mathbb{R}_+^{\mathcal{S}(S)} \to \mathbb{PR}_+^{\mathcal{S}(S)}$$

is an embedding (point is: remains injective after scaling)

Identifying $\mathcal{T}(S)$ & $\mathcal{PML}(S)$ with images in $\mathbb{PR}_+^{\mathcal{S}(S)}$,

have $\mathcal{PML}(S) = \overline{\mathcal{T}(S)} \setminus \mathcal{T}(S) \cong \text{Sphere}^{6g-7+2n}$

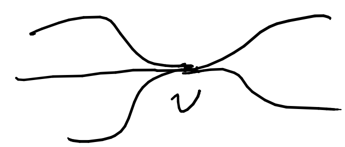
Thus view $\mathcal{PML}(S)$ as <u>boundary</u> of $\mathcal{T}(S)$.

⊛ $\mathrm{Mod}(S) \curvearrowright \mathcal{PML}(S)$, & this action is equivariant wrt viewing $\mathcal{PML}(S)$ as compactification of $\mathcal{T}(S)$

Train Tracks and parametrizing $\mathcal{ML}$

Let S be a surface. A train <u>train track</u> in S is a C^1-embedded graph $\tau \subset S$, satisfying:

- edges called "branches"
- vertices called "switches"
- for each switch v, all branches incident at v share a common tangent line $L_v \le T_v S$
  - ↳ thus branches incident at v divided into 2 sides.



- each component of $S \setminus \tau$ has negative <u>Euler index</u>:

  C comp $S \setminus \tau$, closure $\bar{C}$ is surface with some # $k \geq 0$ of cusps along boundary.

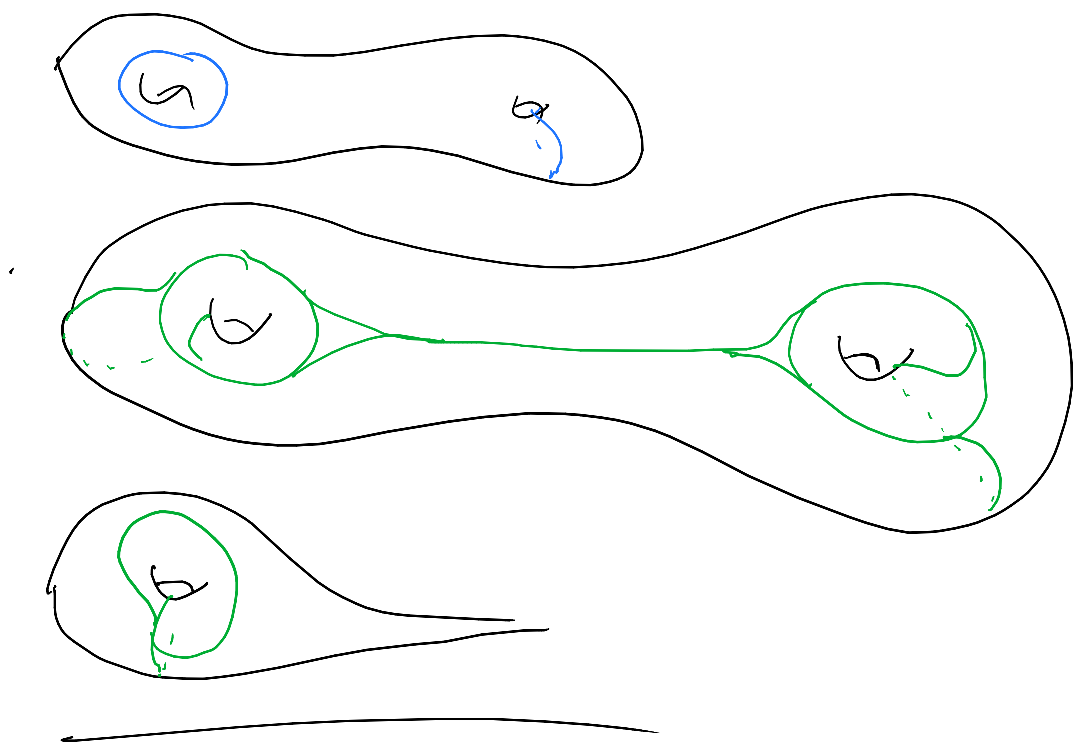
  Euler index $= \chi(C) - k/2$

  $= \frac{1}{2}\chi$ (double of C along boundary, where cusps $\rightsquigarrow$ punctures)

↳ This rules out

0, 1, 2-gons, annuli, & once-punctured monogons

Ex. any multicurve counts as a train track:



Tie Neighborhood of a track τ:

Small nbhd $N \subset S$ of τ equipped with retraction $N \to \tau$ whose fibers give foliation of N by "ties" transverse to τ

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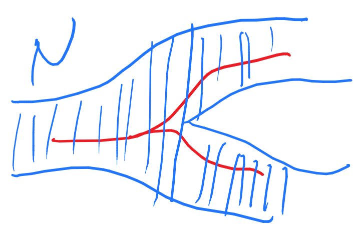
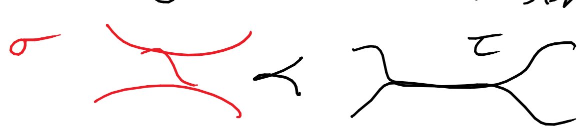
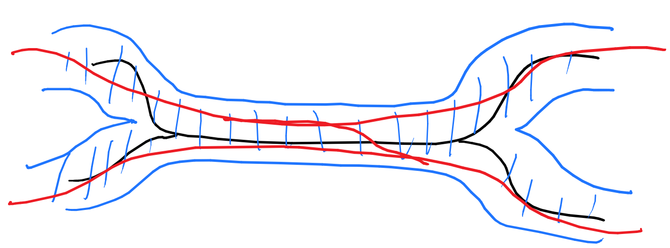


Carrying: σ, τ 2 tracks, say $\sigma < \tau$ (τ carries σ) if σ may be smoothly isotoped into N (= Tie nbhd τ) remaining transverse to the ties:





Train Paths a *train path* on $\tau \subset S$ is a

C^1 path interval $\to \tau$, or $S^1 \to \tau$

equiv: a smooth path into tie nbhd that is transverse to ties

Ex If σ is multicurve (thought of as train track), Then $\sigma \prec \tau \iff$ each component of σ is smoothly isotopic to a train path.

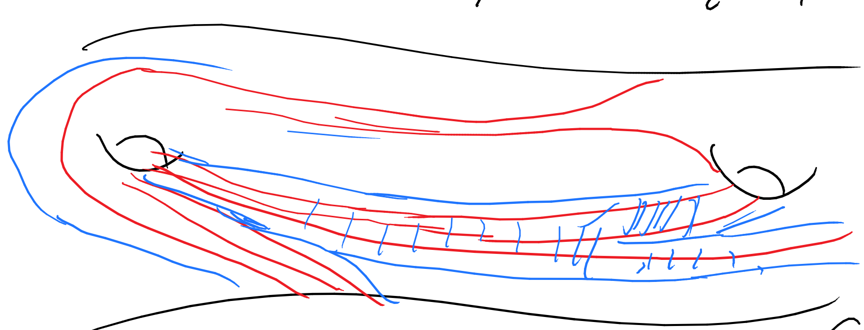
Def Let L be a geodesic Lamination on X, τ a train track.

L *carried* on τ ($L \prec \tau$), if L may be smoothly isotoped into tie nbhd, remaining transverse to ties.

(So: each leaf of L is a train path!)

Fact: Every compact geodesic Lamination $\lambda \subset X$ is carried on some train track:

Pf: for small $\varepsilon > 0$, let $N_\varepsilon \subset X$ be an ε-nbhd of λ.



This "looks" like a train track (C^1-bdry with finitely many cusps).

$X \setminus N_\varepsilon$ has same topology (components, spikes, etc) as $X \setminus \lambda$

May foliate N_ε by arcs transverse to λ

Collapsing these ties gives track τ whose tie nbhd is N_ε

(hence $\lambda < \tau$)

<u>Note:</u> complementary cmps of $X \setminus \tau$ corresp to cmps of $X \setminus \lambda$.

Thus fact that leaves of λ are geodesic

$\Rightarrow$ all cmps of $X \setminus \tau$ have negative Euler index. $\square$

Transverse Weights

Consider train track τ on S. Let B_τ = set of branches of τ.

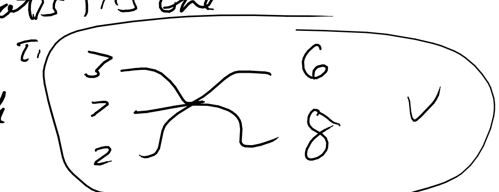
A <u>weight function</u> (transverse weight) on τ is a function

$w : B_\tau \to \mathbb{R}_{\geq 0}$ that satisfies the

<u>switch condition:</u> for every switch σ

$$\boxed{\sum_{\substack{b \text{ incident on} \\ \text{left of } \sigma}} w(b) = \sum_{\substack{b \text{ incident on} \\ \text{right of } \sigma}} w(b)}$$

e.g.



Write $W_\tau \subset \mathbb{R}^{B_\tau}$ for the set of weights on τ

- linear subspace of positive quadrant $\mathbb{R}^{B_\tau}_{\geq 0}$.

<u>Def</u> A train track is <u>recurrent</u> if for every branch $b \in B_\tau$, there exists a closed train path $S^1 \to \tau$ that traverses b.

Prop τ is recurrent $\iff$ τ carries a weight $w \in W_\tau$ with $w(b) > 0 \ \forall b \in B_\tau$.

pf: $\Rightarrow$ let $S^1 \xrightarrow{\alpha} \tau$ be a train path traversing b.

def w_α weight by $w_\alpha(b') = \#$ times α crosses b'

then w_α clearly a weight fnct & $w_\alpha(b) > 0$

take such w_α for each branch b,

add them up $\Rightarrow$ positive weight funct ✓

$\Leftarrow$ Inductive argument: If τ has 1 branch, then $\tau =$ closed curve & its obvious.

Suppose proved for all tracks with $< b$ branch.

let τ be track with τ branchs & $w \in W_\tau$ a positive weight.

find a closed train path $c : S^1 \to \tau$

(can always find one: just set train going, must repeat a branch!)

may assume c traverse each branch ≤ 1 in each dir

let $\varepsilon = \min \{w(b)\}$ st c traverses b

(or $2w(b)$ if traverse both dirs.)

now, consider weight εw_c weight from c

& subtract: $\underline{w - \varepsilon w_c}$: still a weight (non-neg)

- pass to subtrack where supported.

induction $\Rightarrow$ can find closed train path over every remaining branch. $\boxtimes$

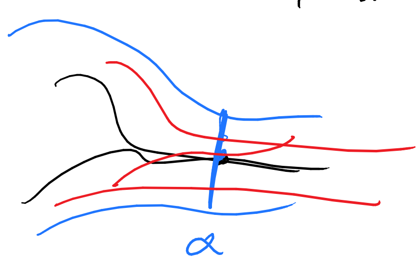
suppose τ carries σ: $\sigma \prec \tau$

choose $\Phi: S \to S$ isotopic to Id ("supporting map" for the carrying)

s.t. $\Phi(\sigma) \subseteq$ Tie nbhd τ, ⋔ ties.

get $|B_\tau| \times |B_\sigma|$ carrying matrix M:

as follows: each branch $b \in B_\tau$, fix tie α_b over interior pt of b



for $b' \in B_\sigma$, set

$$\boxed{M_{(b,b')} = \#\text{ times } \Phi(b') \text{ intersects } \alpha_b}$$

<u>Claim</u>: mult. $M: \mathbb{R}^{B_\sigma} \to \mathbb{R}^{B_\tau}$ restricts to <u>linear map</u>

$W_\sigma \to W_\tau$ (depends on choice of Φ)

(<u>proof</u>: $\Phi(\sigma)$ in tie nbhd ⋔ ties $\Rightarrow$ switch condition preserved)

<u>Eg</u>: If γ scc carried on τ, get 1-dim subspace of W_τ.

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<u>Observe</u>: Suppose μ measured lamination st $\mathrm{Supp}(\mu) = \lambda_\mu \prec \tau$.

Then μ induces weight on τ:

fix central tie α_b over interior pt of branch b.

Set $w_\mu(b)$ = integral measure μ assigns to α_b.

invariance of measure $\Rightarrow$ result satisfies switch conditions.

<u>Fact</u>: this weight is well-defined (indep of carrying)

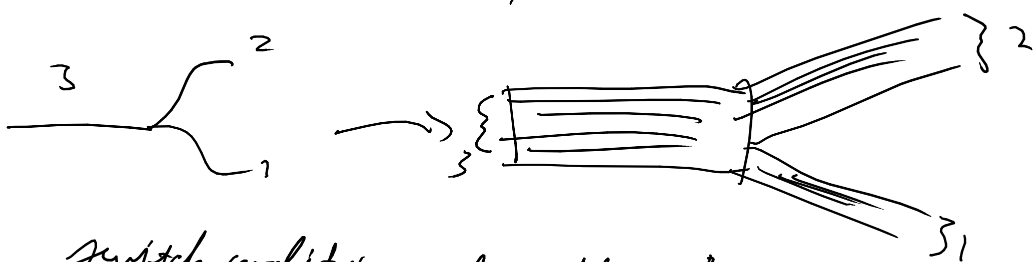
— say measured lam μ <u>carried</u> on τ with weight w_μ.

__Conversely__: Every weighted train track τ, $w \in W_\tau$

defines a measured lamination μ:

__sketch__: for each branch b of τ, take "foliated rectangle" height $w(b)$

, piece together in structure of τ:



switch condition: ends match up! get foliation $\mathcal{F}$ of subset of S

fix hyp metric X on S. $\rightsquigarrow$ pull each leaf of $\mathcal{F}$ tight to __geodesic__ (lift to univ. cover, pull tight there)

Let $\lambda = \bigcup$ all such geodesics

give λ transverse measure st arcs crossing τ get correct weights. ☒

obviously it is technical to do this carefully!

This gives a linear map $\varphi_\tau : W_\tau \to \mathcal{ML}(S)$

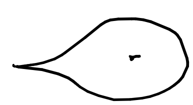
· φ_τ is a continuous injection

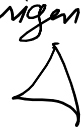
call image $V(\tau) = \varphi_\tau(W_\tau) \subset \mathcal{ML}(S)$.

– this consists of all measured laminations carried on τ.

Thus, φ_τ gives homeo between weights on τ & measured laminations carried on τ.

__Def__ say τ is __maximal__ if any component of $S \setminus \tau$ is a trigon or once-punctured monogon





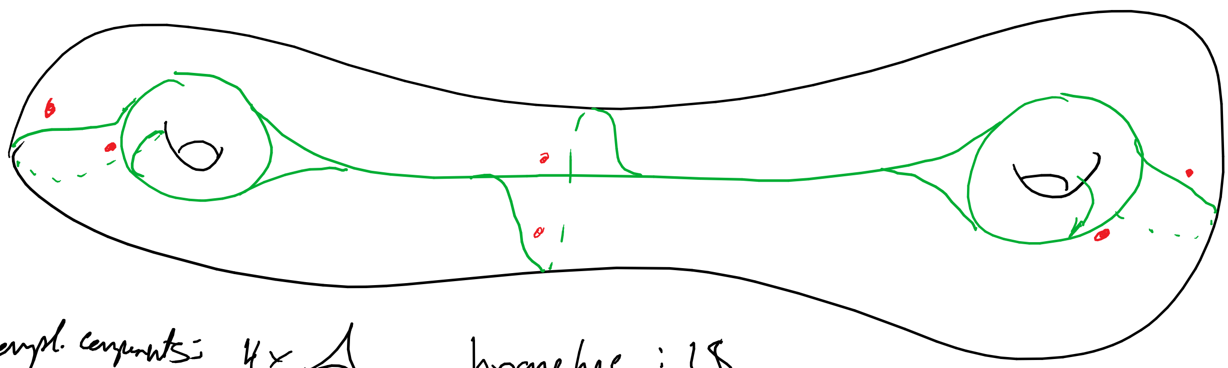
· clear: any track may be extended to a maximal one by adding branches.

In fact: any recurrent track may be extended to maximal recurrent track.

Exercise: If τ is a maximal recurrent track on S, then

$$\dim(W_\tau) = 6g-6+2n.$$

(Euler char stuff; # switches, branches, etc, & # independent switch conditions)



Compl. components: $4\times\triangle$ branches: 18

switches: 12   (12 small, 6 large)

any weight funct. determined by weight on 6 marked * branches.

Thm The space $\mathcal{ML}(S)$ is a piecewise linear manifold of dim $6g-6+2n$, with PL structure defined by charts

$$\Phi_\tau : W_\tau \to V(\tau) \subset \mathcal{ML}(S)$$

for τ a maximal, recurrent train track.

Pf: Every lamination carried on some track, can always extend to maximal. Hence suffices to look at maximal guys, & the images $\{V(\tau)\}$ cover.

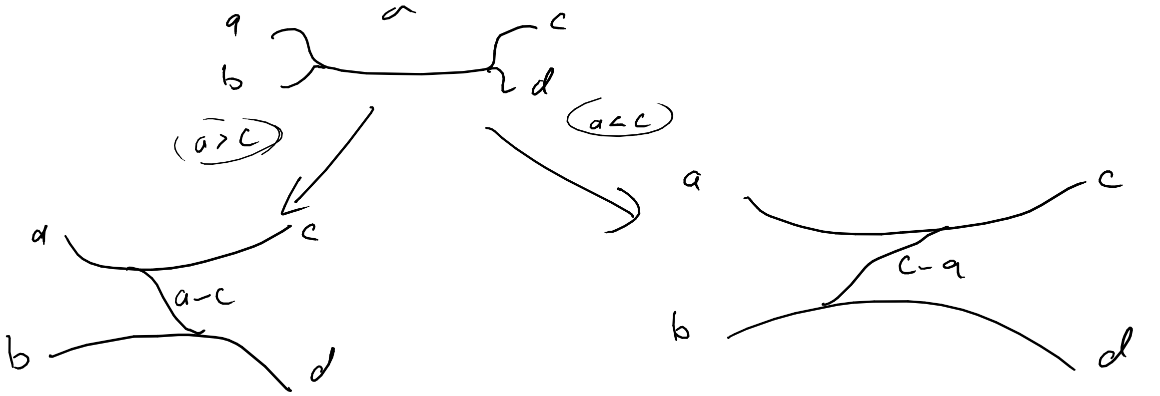
ϕ_τ injective embeddings so images open & we get dimensions.

Key point: If 2 charts $V(\tau)$ & $V(\sigma)$ overlap, then the transition map is piecewise linear:

Idea: choose $\mu \in V(\tau) \cap V(\sigma)$.

μ determines weights on σ.

now split σ in direction determined by these weights:



get σ': new track with weight funct s.t. $\sigma' < \sigma$

repeat, eventually get σ_0 w/ weight defining μ s.t.

$\sigma_0 < \sigma$ and $\sigma_0 < \tau$

carrying maps give linear inclusions $W_{\sigma_0} \hookrightarrow W_\sigma \cong V(\sigma)$, $W_{\sigma_0} \hookrightarrow W_\tau \cong V(\tau)$

So: In nbhd of $\mu \in U(\tau)$,

transition is linear ∎

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Furthermore: The linear transition functs are in $SL(d, \mathbb{R})$

Hence preserve the standard volume form on $W_\tau \subset \mathbb{R}^{B_\tau}$. Thus

Thm (Thurston) There is a canonical measure μ_{Th} ("Thurston measure") on $\mathcal{ML}(S)$ coming from the PL-structure.
Has property: for any $A \subset \mathcal{ML}(S)$ measurable,

$$\mu_{Th}(hA) = h^{6g-6+2n}\mu_{Th}(A). \quad ∎$$

(In fact: μ_{Th} is measure associated to a symplectic structure on $\mathcal{ML}$)

need fact about μ_{Th}:

<u>Thm</u> (Masur)

Thurston measure μ_{Th} is ergodic for the action of $\mathrm{Mod}(S)$ on $\mathcal{ML}(S)$.

for any $\mathrm{Mod}(S)$-invariant set $A \subset \mathcal{ML}(S)$, have either $\mu_{Th}(A) = 0$ or

$$\mu_{Th}(\mathcal{ML}(S) \setminus A) = 0.$$