Mon 2/26/18

# V - The Volume Recursion

We now shift toward using the McShane-Mirzakhani identity to establish the volume recursion & polynomial nature of the volumes.

Henceforth, given $L = (L_1, \dots, L_n)$, $L_i \geq 0$, write

$$\mathcal{T}_g(L) = \mathcal{T}_g(L_1, \dots, L_n) \qquad \text{assume } 3g-3+n>0$$

= Teichmüller space of hyperbolic genus $g$ surfaces w/ geodesic boundary of lengths $L_1, \dots, L_n$

($L_i = 0 \Leftrightarrow$ puncture/cusp)

equiv: $S$ = genus $g$ surf with $n$ bdry comps ($L_i > 0$)/punctures ($L_i = 0$)

$$\mathcal{T}_g(L) = \{(X,f) \mid X \text{ as above}, f: S \to X \text{ homeo}\}/\sim$$

$(X,f) \sim (Y,g)$ if $g \circ f^{-1}: X \to Y$ homotopic to isometry.

* homotopies must preserve boundary / punctures setwise. need not fix bdry pointwise.

If $R$ hyp surf, also write $\mathcal{T}(R)$ for $\mathcal{T}_g(\ell_{\beta_1}(R), \dots, \ell_{\beta_n}(R))$ where $R$ has genus $g$ & bdry/punctures $\beta_1, \dots, \beta_n$.

$MCG = \mathrm{Mod}_g(L) = \mathrm{Homeo}^+(S)/\text{homotopy}$.

$\mathrm{Mod}_g(L) \curvearrowright \mathcal{T}_g(L)$ via $\phi \cdot (X,f) = (X, f \circ \phi^{-1})$

moduli space $\mathcal{M}_g(L) = \mathcal{T}_g(L)/\mathrm{Mod}_g(L)$

Have Fenchel-Nielsen coords on $\mathcal{T}_g(L)$ just as before:

- pants decomp has $3g-3+n$ essential curves
- each gives length $l_i$ & twist $\tau_i$ params.

(don't use bdry curves; their lengths fixed!)

$\mathcal{T}_g(L) \xrightarrow{\cong} (\dots, l_i, \dots, \tau_i, \dots)$ homeo.

Have WP symplectic form $\omega = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i$

- invariant under $Mod_g(L)$

$\rightsquigarrow$ induces symplectic form + assoc. volume form on $\mathcal{M}_g(L)$.

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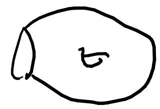
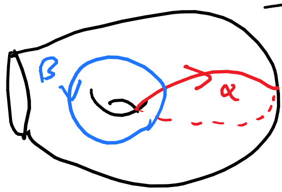
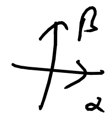
# Half Twists

Recall: each essential simple closed curve $\gamma \subset S$
has Dehn twist $D_\gamma \in Mod_g(L)$.

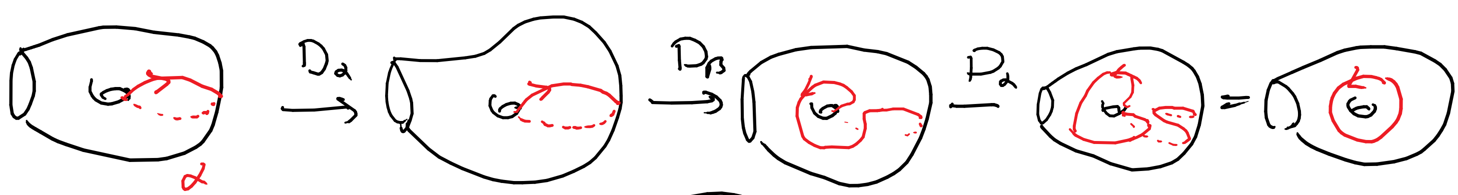
If $\gamma$ bounds a torus ( one comp of $S \setminus \gamma$ is $\cong$  $= S_1^1$ )
there is also a __half twist__ about $\gamma$ defined as follows:

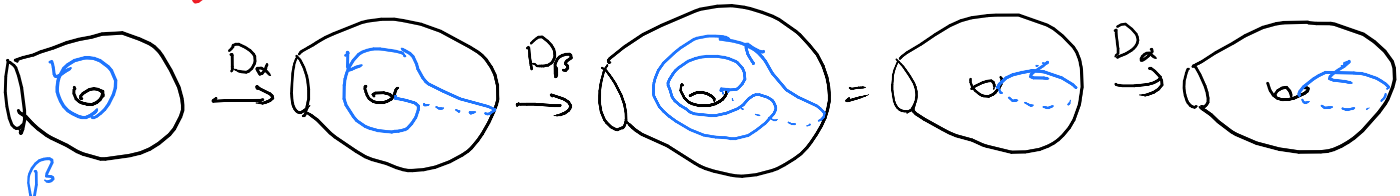



Take oriented curves $\alpha$ & $\beta$ as pictured,
so 

is pos orientation on surf.

Consider: $\phi = D_\alpha \circ D_\beta \circ D_\alpha$.



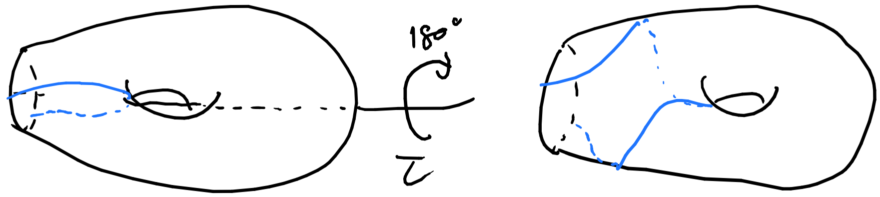




__Exercise:__ $\phi = D_\alpha D_\beta D_\alpha = D_\beta D_\alpha D_\beta$ (they agree on all curves)

So, $\phi(\alpha) = \beta$, $\phi(\beta) = \bar{\alpha}$ $\Rightarrow$ $\left.\begin{array}{l}\phi^2(\alpha) = \bar{\alpha} \\ \phi^2(\beta) = \bar{\beta}\end{array}\right\}$ $\phi^2$ fixes every essential curve of $S_1^1$, but reverses its orientation.

Thus $(\phi^2)^2$ fixes $\alpha, \beta$ ptwise : cut on $\alpha$ + $\beta$ $\rightsquigarrow$
have annulus fixing 1 bdy ptwise $\rightsquigarrow$ isotopic to Id ( Alexander Lemma )
so $(\phi^2)^2 = 1 \in Mod(S_1^1)$.

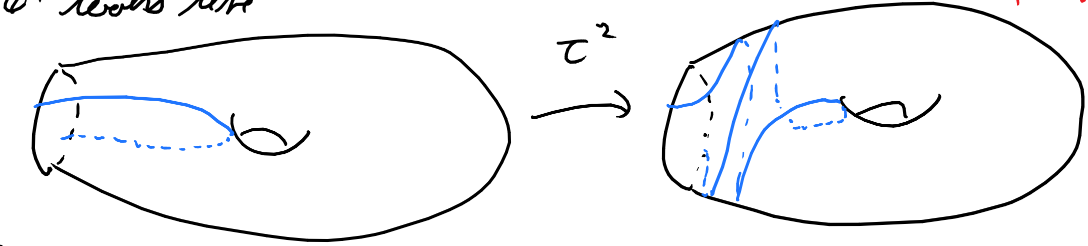
Notice:




Note: $\phi^2 = \tau$ use alexander Lemma)

$\phi^2 = \tau$ acts trivially on $\mathcal{T}(S_1^1)$ $\Longrightarrow$ formally divide $V_1(L_1)$ by 2

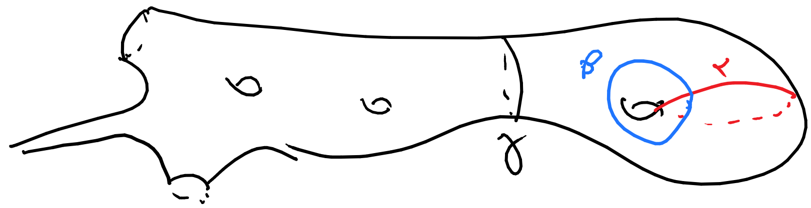
So $\tau^2 = \phi^4$ looks like




Now let $S$ be arbitrary, & $\gamma \subset S$ scc bounding a torus. Pick $\alpha, \beta$ as before.

Then

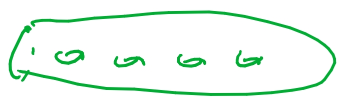
$(D_\alpha D_\beta D_\alpha)^4$ = Dehn twist $D$




so $(D_\alpha D_\beta D_\alpha)^2$ = half twist about $\gamma$

$= H_\gamma$ "$= D_\gamma^{1/2}$"

Thus $H_\gamma^2 = D_\gamma$

If $\gamma$ separating & bounds a surface genus $>1$, there is still a square root of $D_\gamma$: just rotate that surf. by 180°. But this is not canonical! have to choose how to draw complement as  & thus what rotate means. In case of torus there is a canonical way to do this, so we may safely define the half twist $H_\gamma$.

In FN coords in which $\gamma$ = the $i^{th}$ pants curve

The half twist fixes $l_j$ & $\tau_j$ $\forall j \neq i$ & acts as $(l_i, \tau_i) \longmapsto (l_i, \tau_i + l_i/2)$

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## Setup for covolume formula:

$\mathcal{T}(R) = \mathcal{T}_g(l_{\beta_1}(R), \dots, l_{\beta_n}(R))$ associated Teich space

$R$ hyp surf with geodesic boundary $\beta_1, \dots, \beta_n$,

Let $\gamma_1, \dots, \gamma_m$ be distinct, disjoint simple closed geodesics.

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Consider <u>weighted multicurve</u> $\gamma = \sum_{i=1}^{m} a_i \gamma_i$, where $a_i \in \mathbb{R}$

$\mathrm{Mod}(R) = \mathrm{Mod}(S)$ acts on the set of weighted multicurves.
↳ preserve Bdry comps!

Let $\mathrm{Stab}(\gamma) \leq \mathrm{Mod}(R)$ be stabilizer of $\gamma$ (so $\phi \in \mathrm{Stab}(\gamma)$ may permute components of $\gamma$ that have the same weights)

Let $\mathrm{Stab}_0(\gamma_i) \leq \mathrm{Stab}(\gamma_i)$ = stabilizer of curve $\gamma_i$
" subgroup preserving orientation of $\gamma_i$

have homom $\mathrm{Stab}(\gamma) \longrightarrow$ permutation group of set $\{\gamma_1^+, \gamma_1^-, \dots, \gamma_m^+, \gamma_m^-\}$

image is $\cong \mathrm{Sym}(\gamma) = \mathrm{Stab}(\gamma) / \bigcap_j \mathrm{Stab}_0(\gamma_j)$
= Symmetry group of $\gamma$

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$\boxed{\text{Wed 2/28/18}}$

Write $R(\gamma) = R \setminus \bigcup_i \gamma_i$ cut open surf. (may be disconnected)

each $\gamma_i$ gives rise to 2 boundary components of $R(\gamma)$

for $x = (x_1, \dots, x_m) \in \mathbb{R}^m_{\geq 0}$

- $\mathcal{T}(R(\gamma); x)$ = (product) Teich space of top surf $R(\gamma)$ where 2 bdry comps corresp to $\gamma_i$ have length $x_i$, + original boundary components $\beta_1, \dots, \beta_n$ of $R$ have same lengths as in $R$.
- $\mathrm{Mod}(R(\gamma))$ = prod. Mapping class groups of components of $R(\gamma)$
- $\mathcal{T}(R(\gamma); x) / \mathrm{Mod}(R(\gamma))$ = corresp. product of moduli spaces
- $V(R(\gamma); x)$ = product of component moduli spaces
↳ factors for components $R' \cong S_1^1$ of $R(\gamma)$, $V(R') = \frac{1}{2}$ true volume.

Recall, $S = R$ our ref surf, each scc $\alpha$ on $S \longrightarrow$ length function

$$\ell_\alpha : \mathcal{T}_g(L_1, \dots, L_n) \to \mathbb{R}_+, \quad R \mapsto \ell_\alpha(R)$$

$\phi \in \mathrm{Mod}$ acts on length functions $\ell_\gamma$

$\boxed{5\text{-}4}$ $\ell_\alpha \circ \phi^{-1} = \ell_{\phi(\alpha)}$ : $\ell_\alpha(\phi^{-1}(R)) = \ell_{\phi(\alpha)}(R)$

Our multicurve $\gamma = \sum_{j=1}^m a_j \gamma_j \rightsquigarrow$ length fcn

$$\underline{l_\gamma = \sum_{i=1}^m a_i l_{\gamma_i}}$$

Given a function $f: \mathbb{R} \to \mathbb{R}_+$ (suitably small at $\infty$ ... so that things converge!)

Notice that the function

$$f \circ l_\gamma : \mathcal{T}_g(L) \longrightarrow \mathbb{R}_+$$
$$R \mapsto f(l_\gamma(R)) = f\left(\sum_{j=1}^m a_j l_{\gamma_j}(R)\right)$$

is invariant under action of $\mathrm{Stab}(\gamma) \curvearrowright \mathcal{T}_g(L)$

+ the function $f_\gamma : \mathcal{T}_g(L) \to \mathbb{R}_+$ , $\underline{f_\gamma = \sum_{\mathrm{Mod}/\mathrm{Stab}(\gamma)} f \circ l_\gamma \circ h^{-1}}$

$$f_\gamma(R) = \sum_{h \in \mathrm{Mod}(R)/\mathrm{Stab}(\gamma)} f\left(l_{h(\gamma)}(R)\right)$$

is invariant under full action of $\mathrm{Mod}(R)$;

$\Rightarrow$ descends to a funct. on $\mathcal{M}(R) = \mathcal{T}(R)/\mathrm{Mod}(R)$

Now express integral of $f_\gamma$ as weighted integral over lower-dim moduli spaces.

<u>Covolume Formula (Mirzakhani)</u> given multicurve $\gamma = \sum_{i=1}^m a_i \gamma_i$

$$\int_{\mathcal{T}(R)/\mathrm{Mod}(R)} f_\gamma \, dV = \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{\mathbb{R}^m_{>0}} f(|x|)\, V(R(\gamma); x)\, x \cdot dx$$

where $x = (x_1, \dots, x_m)$, $|x| = \sum_{i=1}^m a_i x_i$, $x \cdot dx = x_1 \cdots x_m \, dx_1 \cdots dx_m$

<u>proof</u>: Have

$$\sum_{h \in \mathrm{Mod}/\bigcap \mathrm{Stab}_0(\gamma_i)} f \circ l_\gamma \circ h^{-1} = \sum_{\mathrm{Mod}/\mathrm{Stab}(\gamma)} \overbrace{\sum_{\mathrm{Stab}(\gamma)/\bigcap \mathrm{Stab}_0(\gamma_i)} f \circ l_\gamma \circ h^{-1}}^{f \circ l_\gamma \text{ inv't under } \mathrm{Stab}(\gamma)!} = |\mathrm{Sym}(\gamma)|\, f_\gamma$$

So:

$$\int_{\mathcal{T}(R)/\mathrm{Mod}(R)} f_\gamma \, dV = \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{\mathcal{T}(R)/\mathrm{Mod}(R)} \sum_{h \in \mathrm{Mod}(R)/\bigcap \mathrm{Stab}(\gamma_i)} \underbrace{f \circ \ell_\gamma \circ h^{-1}}\, dV$$

recall $f \circ \ell_\gamma$ invt under $\mathrm{Stab}(\gamma) \geq \bigcap \mathrm{Stab}_0(\gamma_i)$

$$= \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{\mathcal{T}(R)/\bigcap \mathrm{Stab}_0(\gamma_i)} f \circ \ell_\gamma \, dV$$

have short exact sequence:

$$1 \to \langle D_{\gamma_1}, \dots, D_{\gamma_m} \rangle \to \bigcap \mathrm{Stab}_0(\gamma_i) \to \overbrace{\prod_{R' \text{ cmp } R(\gamma)} \mathrm{Mod}(R')}^{\mathrm{Mod}(R(\gamma))} \to 1$$

$\langle D_{\gamma_1}, \dots, D_{\gamma_m} \rangle \overset{?}{\cong} \mathbb{Z}^m = \prod \mathrm{Dehn}(\gamma_i)$

(half-twists for $\gamma_i$ bounding torus live here)

+ associated fibration of Teich spaces coming from FN coords:

$$\begin{array}{ccc} \mathcal{T}(R(\gamma); x) & \hookrightarrow & \mathcal{T}(R) \\ \| & & \downarrow \\ \prod_{R' \text{ cmp } R(\gamma)} \mathcal{T}(R') & & \prod_{\gamma_j} \mathbb{R}_{>0} \times \mathbb{R} \end{array}$$

$$\boxed{dV(R) = \prod_{R' \text{ cmp } R(\gamma)} dV(R') \times \prod_{\gamma_j} d\ell_j \wedge d\tau_j}$$

★ Wolpert formula ⟹

- a fibration of symplectic mnflds
- vol form on $\mathcal{T}(R)$ is locally a product w.r.t. this fibration.

Descends to fibration of moduli spaces:

$$\begin{array}{ccc} \mathcal{M}(R(\gamma); x) & \hookrightarrow & \mathcal{T}(R) / \bigcap_j \mathrm{Stab}_0(\gamma_j) \\ \| & & \downarrow \\ \prod_{R' \text{ cmp } R(\gamma)} \mathcal{T}(R')/\mathrm{Mod}(R') & & \prod_{\gamma_j} (\mathbb{R}_{>} \times \mathbb{R}) / \overline{\mathrm{Dehn}_*}(\gamma_j) \end{array}$$

$\mathrm{Dehn}_*(\gamma_j)$ gen by half twist $H_{\gamma_j}$ if $\gamma_j$ bounds torus $(\ell_j, \tau_j) \Rightarrow (\ell_j, \tau_j + \frac{\ell_j}{2})$ or Dehn twist else: $(\ell_j, \tau_j) \Rightarrow (\ell_j, \tau_j + \ell_j)$

Thus

$$\int_{\mathcal{T}(R)/\mathrm{Mod}(R)} f_\gamma \, dV = \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{\mathcal{T}(R)/\bigcap_j \mathrm{Stab}_0(\gamma_j)} f \circ l_\gamma \, dV$$

function only depends on value $|x|$ !

$$= \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{(x,\tau) \in \prod_j \mathbb{R}_{>0} \times \mathbb{R} / \mathrm{Dehn}_x(\gamma_j)} \int_{\mathcal{T}(R(\gamma);x)/\mathrm{Mod}(R(\gamma))} f \circ l_\gamma$$

$$= \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{(x_1,\dots,x_m) \in \mathbb{R}_{>0}^m} \int_{\substack{0 \le \tau_i \le x_i \\ \text{or} \\ 0 \le \tau_i \le x_i/2}} \int_{R' \in \mathcal{T}(R(\gamma);x)/\mathrm{Mod}(R(\gamma))} f(|x|) \, dV(R') \, d\tau \, dx$$

$$= \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{x \in \mathbb{R}_{>0}^m} f(|x|) \int_{R' \in \mathcal{T}(R(\gamma);x)/\mathrm{Mod}(R(\gamma))} dV(R') \frac{1}{2^{\#(\gamma_i \text{ bounding tori})}} x_1 \cdots x_m \, dx_1 \dots dx_m$$

$$= \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{x \in \mathbb{R}_{>0}^m} f(|x|) \, \mathrm{Vol}(R(\gamma);x) \, x \cdot dx \qquad \blacksquare$$

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## Volume Recursion Thm (Mirzakhani)

The WP volume $V_g(L_1,\dots,L_n)$ of the moduli space $\mathcal{T}_g(L_1,\dots,L_n)/\mathrm{Mod}_g(L_1,\dots,L_n)$ is a symmetric function of boundary lengths $L_1,\dots,L_n$ defined recursively as follows:

For $L_1,L_2,L_3 \geq 0$ formally set

- $V_0(L_1,L_2,L_3)=1$
- $V_1(L_1) = \frac{\pi^2}{12} + \frac{L_1^2}{48}$ $\left(= \frac{1}{2} \text{ true value of } V_1(L_1)\right)$

For $L=(L_1,\dots,L_n)$ & $(g,n)\neq(0,3)$ or $(1,1)$, volume satisfies:

$$\frac{\partial}{\partial L_1} L_1 V_g(L) = A_g^{con}(L) + A_g^{dis}(L) + B_g(L), \text{ where:}$$

$$A_g^*(L) = \frac{1}{2}\int_0^\infty\int_0^\infty \hat{A}_g^*(x,y,L)\, xy\, dxy$$

$$B_g(L) = \int_0^\infty \hat{B}_g(x,L)\, x\, dx \quad \& \quad \hat{A}_g^{con}, \hat{A}_g^{dis}, \hat{B}_g \text{ given by:}$$

$$\hat{A}_g^{con}(x,y,L) = H(x+y,L_1)\, V_{g-1}(\hat{L}), \qquad \hat{L} = (L_2,\dots,L_n)$$

$$\hat{A}_g^{dis}(x,y,L) = \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\dots,n\}}} H(x+y,L_1)\, V_{g_1}(x,L_{I_1})\, V_{g_2}(y,L_{I_2})$$

giving hyp structures?

omit

$$\hat{B}_g(x,L) = \frac{1}{2}\sum_{j=2}^{n}\left(H(x,L_1+L_j) + H(x,L_1-L_j)\right) V_g(x,L_2,\dots,\hat{L}_j,\dots,L_n)$$

$$\& \quad H(x,y) = \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}$$

Point: $V_g(L_1,\dots,L_n)$ is an integral over volumes of lower complexity surfaces

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Proof: $\cdot$ $V_0(L_1,L_2,L_3) = 1$ since $\mathcal{M}_0(L_1,L_2,L_3)$ consists of 1 point.

$\cdot$ Lets come back to $V_1(L_1)$

For the recursion, know $\forall x \in \mathcal{M}_g(L_1,\dots,L_n)$

$$L_1 = \underbrace{\sum_{(\alpha_1,\alpha_2)\in \mathcal{J}_1} D(L_1, l_{\alpha_1}(x), l_{\alpha_2}(x)) + \sum_{j\geq 2}^{n} \sum_{\gamma \in \mathcal{J}_{1j}} R(L_1, L_j, l_\gamma(x))}_{\text{constant function on } \mathcal{M}_g(L).}$$

so $L_1 V_g(L) = \int_{\mathcal{M}_g(L)}$ $\nearrow$ this.

- write each sum as sum over MCG-orbit of a multicurve + apply covolume formula.
- take $\frac{\partial}{\partial L_1}$ derivative to simplify functions $D$ + $R$:

$$\frac{\partial}{\partial L_1} D(L_1, x, y) = H(x+y, L_1)$$

$$\frac{\partial}{\partial L_1} R(L_1, L_j, x) = \frac{1}{2}\Big(H(x, L_1+L_j) + H(x, L_1 - L_j)\Big)$$

$\cdot$ orbits of $(\alpha_1,\alpha_2) \in \mathcal{J}_1$ : 2 possibilities:

1) $R \setminus (\alpha_1 \cup \alpha_2)$ connected;


There is one MCG orbit of such un-ordered pairs! (classification of surfaces principle)

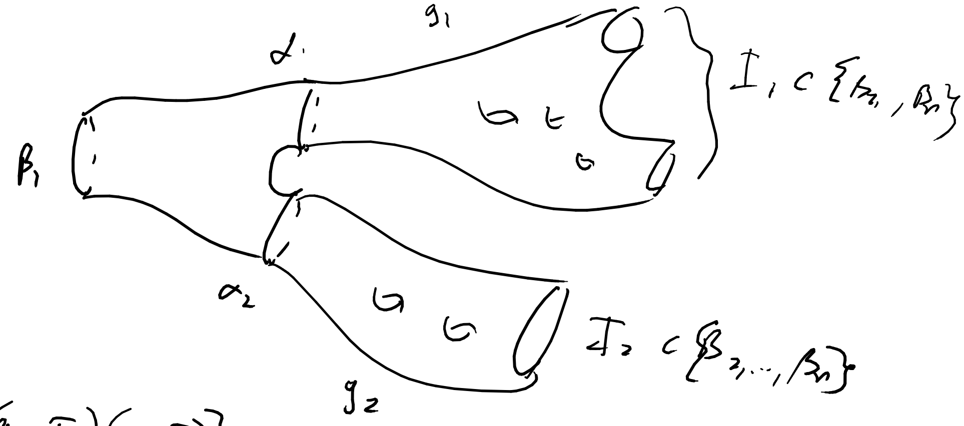
genus $g-1$
boundary $\alpha_1, \alpha_2, \beta_2, \dots, \beta_n$

— recall that in Mod we require elements to preserve each boundary component. Hence if $(\alpha_1,\alpha_2)\in \mathcal{J}_1$, then $\phi(\alpha_1,\alpha_2) \in \mathcal{J}_1$, $\forall \phi$.

Fix some such $(\delta_1^0, \delta_2^0) \in \mathcal{J}_1$, + set $\gamma_0 = \delta_1^0 + \delta_2^0$ weighted multicurve. $|\mathrm{Sym}(\gamma_0)| = 2$

2) $R \setminus (\alpha_1 \cup \alpha_2)$ disconnected.

one MCG orbit for each topological way to disconnect surface.




That is: Let $\mathcal{I}_{g,n} = \{(g_1, I_1), (g_2, I_2)\}$

s.t.:

$g_1, g_2 \geq 0, \quad g_1 + g_2 = g$

$\{2, \dots, n\} = I_1 \sqcup I_2$

$2 \leq 2g_1 + |I_1|$

$2 \leq 2g_2 + |I_2|$

$g = g_1 + g_2$

$I_1 \sqcup I_2 = \{2, \dots, n\}$

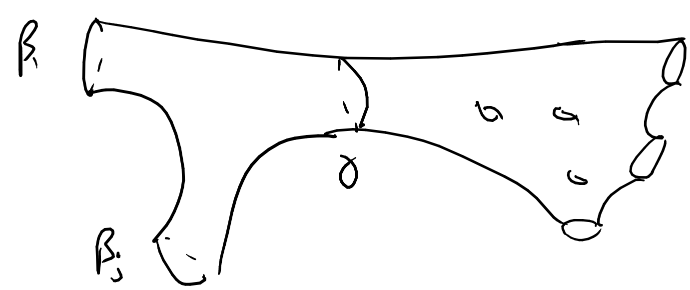
Then $\mathcal{I}_{g,n} \xleftrightarrow{\text{bijection}}$ MCG-orbits of $\{(\alpha_1, \alpha_2) \in \mathcal{J}_1 \mid R \setminus (\alpha_1 \cup \alpha_2) \text{ disconnected}\}$

– the partition of other boundary comps is preserved by MCG elts.

For each $a \in \mathcal{I}_{g,n}$ partition, fix some $(\delta_1^a, \delta_2^a) \in \mathcal{J}_1$ s.t. st $R \setminus (\delta_1^a \cup \delta_2^a)$ = pants $\sqcup$ surg genus $g_1$ w/ bdry $I_1$ $\sqcup$ genus $g_2$ w bdry $I_2$.

Set $\gamma_a = \delta_1^a + \delta_2^a$ weighted multicurve. $|\mathrm{Sym}(\gamma_a)| = 1$

- For each $j \neq 1$, there is exactly one MCG-orbit of $\mu \in \mathcal{J}_{1,j}$.




again, use fact that we preserve boundary:
If $\gamma \in \mathcal{J}_{1,j}$, then $\phi\gamma \in \mathcal{J}_{1,j}$

For each $j = 2, \dots, n$, fix some $\gamma_j \in \mathcal{J}_{1,j}$

& consider weighted multicurve $\gamma_j$, $\quad |\mathrm{Sym}(\gamma_j)| = 1$

For each eheyen weighted multicurve $\gamma = \gamma_0, \gamma_a, a \in \mathcal{I}_{g,n}, \gamma_j$,
have corresp funt $f_\gamma$ on $\mathcal{M}_g(L)$:

consider $f(x) = H(x, L_1) = \dfrac{1}{1+e^{\frac{x+L_1}{2}}} + \dfrac{1}{1+e^{\frac{x-L_1}{2}}}$

for $\gamma = \gamma_0$ or $\gamma_a$, $a \in \mathcal{I}_{g,n}$, consider $f \circ l_\gamma = H(l_\gamma, L_1)$

& $f_\gamma = \sum_{h \in \mathrm{Mod}/\mathrm{Stab}(\gamma)} H(l_{h(\gamma)}, L_1)$

Also, for each $j = 2, \dots, n$, consider $f_j(x) = \frac{1}{2} H(x, L_1 + L_j) - \frac{1}{2} H(x, L_1 - L_j)$

have $f_j \circ l_{\gamma_j}$ a funct on $\mathcal{M}(R)$:

$$f_{\gamma_j} = \sum_{h \in \mathrm{Mod}/\mathrm{Stab}(\gamma_j)} \left( \frac{1}{2} H(l_{h(\gamma_j)}, L_1 + L_j) - \frac{1}{2} H(l_{h(\gamma_j)}, L_1 - L_j) \right)$$

Now, have:

$$L_1 V_g(L) = \int_{x \in \mathcal{M}_g(L)} \sum_{(\alpha_1, \alpha_2) \in \mathcal{F}_1} D(L_1, l_{\alpha_1}(x), l_{\alpha_2}(x)) + \sum_{j=2}^{n} \sum_{\gamma \in \mathcal{F}_{1j}} R(L_1, L_j, l_\gamma(x))$$

$$\frac{\partial}{\partial L_1} L_1 V_g(L) = \int_{x \in \mathcal{M}_g(L)} \sum_{(\alpha_1, \alpha_2) \in \mathcal{F}_1} H(l_{\alpha_1}(x) + l_{\alpha_2}(x), L_1) + \sum_{j=2}^{n} \sum_{\gamma \in \mathcal{F}_{1j}} \frac{1}{2} H(l_\gamma(x), L_1 + L_j) + \frac{1}{2} H(l_\gamma(x), L_1 - L_j)$$

$$= \int_{x \in \mathcal{M}_g(L)} f_{\gamma_0} + \sum_{a \in \mathcal{I}_{g,n}} f_{\gamma_a} + \sum_{j=2}^{n} f_{\gamma_j}$$

(apply covolume formula)

$$= \frac{1}{|\mathrm{Sym}(\gamma_0)|} \int_{x,y=0}^{\infty} H(x+y, L_1)\, \overbrace{V_0(x,y,L_1)}^{1}\, V_{g-1}(x,y,L_2,\dots,L_n)\, dx\,dy$$

$$+ \sum_{a\in I_{g,n}} \frac{1}{|\mathrm{Sym}(\gamma_0)|} \int_{x,y=0}^{\infty} H(x+y, L_1)\, V_{g_1}(x, L_{I_1})\, V_{g_2}(y, L_{I_2})$$

$$+ \sum_{j=2}^{n} \frac{1}{|\mathrm{Sym}(\gamma_j)|} \int_{x=0}^{\infty} \frac{1}{2}\big(H(x, L_1+L_j) + H(x, L_1-L_j)\big)\, V_g(x, L_2,\dots,\widehat{L_j},\dots,L_n)$$

as claimed! ∎

**Cor** for each $g,n$, $V_g(L_1,\dots,L_n)$ is a polynomial in $L_1^2,\dots,L_n^2$ with coeff in $\mathbb{Q}[\pi]$

proof: by induction; assume true, then recursion says you get new vol by integrating $V$ against $H$. If smaller Vols are polynomials, recursion comes down to doing 2 integrals:

$$\int_0^\infty x^{2j+1} H(x,t)\,dx \quad \& \quad \int_0^\infty\int_0^\infty x^{2j+1} y^{2j+1} H(x+y, t)\,dx\,dy$$

calculation: each is polynomial in $t^2$ ($t = L_1$ or $L_1 \pm L_j$ in recursion) with coeff. product of factorials and Riem Zeta function $\zeta$(non-neg even integer).

$\Rightarrow$ each coeff is a positive rational multiple of an appropriate power of $\pi$. ∎

Mon 3/12/18

Last time: Volume Recursion formula:

W.P. volume $V_g(L_1,\dots,L_n)$ of moduli space of genus $g$ hyp surfs w/ geodesic boundary of length $L_1,\dots,L_n$ satisfies recursion:

For $L=(L_1,\dots,L_n)$ & $(g,n)\neq(0,3)$ or $(1,1)$, volume satisfies:

$$\frac{\partial}{\partial L_1} L_1 V_g(L) = A_g^{con}(L) + A_g^{dis}(L) + B_g(L), \text{ where:}$$

$$A_g^{*}(L) = \frac{1}{2}\int_0^\infty\int_0^\infty \hat{A}_g^{*}(x,y,L)\, xy\, dx\, dy$$

$$B_g(L) = \int_0^\infty \hat{B}_g(x,L)\, x\, dx \quad \& \quad \hat{A}_g^{con},\ \hat{A}_g^{dis},\ \hat{B}_g \text{ given by:}$$

$$\hat{A}_g^{con}(x,y,L) = H(x+y,L_1)\, V_{g-1}(\hat{L}), \qquad \hat{L} = (L_2,\dots,L_n)$$

$$\hat{A}_g^{dis}(x,y,L) = \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\dots,n\}}} H(x+y,L_1)\, V_{g_1}(x,L_{I_1})\, V_{g_2}(y,L_{I_2})$$

giving hyp structures?

omit

$$\hat{B}_g(x,L) = \frac{1}{2}\sum_{j=2}^{n}\left(H(x,L_1+L_j)+H(x,L_1-L_j)\right)V_g(x,L_2,\dots,\hat{L}_j,\dots,L_n)$$

$$\& \quad H(x,y) = \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}$$

Base case: $V_0(L_1,L_2,L_3) = 1$

$$V_1(L_1) = \frac{\pi^2}{12} + \frac{L_1^2}{48}$$

proof of base case $(g,n)=(1,1)$:

here length identity is: $\forall X \in \mathcal{M}_1(L_1)$,

$$L_1 = \sum_{\substack{\gamma \text{ scc} \\ \text{on } X}} D(L_1, \ell_\gamma(X), \ell_\gamma(X))$$

fix scc $\gamma$ on $S$, get $\ell_\gamma : \mathcal{T}_1(L_1) \to \mathbb{R}_+$

for $f(y) = D(L_1, y, y)$, get

$$f_\gamma = \sum_{h \in \mathrm{Mod}(S)/\mathrm{Stab}(\gamma)} f(\ell_{h(\gamma)}) = \sum f \circ \ell_\gamma \circ h^{-1}$$

defines funct $f_\gamma : \mathcal{M}_1(L_1) \to \mathbb{R}_+$.

Length Id $\Rightarrow f_\gamma(X) = L_1 \quad \forall X \in \mathcal{M}_1(L_1)$, so

$$L_1 V_1(L_1) = \int_{X \in \mathcal{M}_1(L_1)} L_1 \, dV(X) = \int_{\mathcal{M}_1(L_1)} f_\gamma(X) \, dV(X)$$

$$\overset{\text{covol-formula}}{=} \frac{1}{|\mathrm{Sym}(\gamma)|} \int_{\mathbb{R}_+} f(x) \underbrace{V_0(X, x, L_1)}_{1} \, x \, dx$$

$$= \int_{x=0}^{\infty} D(L_1, x, x) \, x \, dx$$

$\Rightarrow$

$$\frac{\partial}{\partial L_1} L_1 V_1(L_1) = \int_{x=0}^{\infty} \frac{\partial}{\partial L_1} D(L_1, x, x) \, x \, dx$$

$$= \int_{x=0}^{\infty} \left( \frac{1}{1+e^{\frac{x-L_1}{2}}} + \frac{1}{1+e^{\frac{x+L_1}{2}}} \right) x \, dx$$

$$= (\text{... power series \& dilogarithms}) = \frac{\pi^2}{6} + \frac{L_1^2}{8}$$

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$$\frac{d}{dL} L V_1(L) = \frac{\pi^2}{6} + \frac{L^2}{8}$$

must have $C=0$

$$L V(L) = \frac{\pi^2}{6} L + \frac{L^3}{24} + C$$

$$\Rightarrow \boxed{V(L) = \frac{\pi^2}{6} + \frac{L^2}{24}}$$

(but in this case we formally divided by 2!) $\boxtimes$

---

As mentioned before, volume recursion shows that vol is a polynomial in $L_1, \dots, L_n$. We will need to know its degree:

**Thm** (Mirzakhani) The volume $V_g(L) = V_g(L_1, \dots, L_n)$ has the form:

$$V_g(L) = \sum_{\substack{\alpha \\ |\alpha| \le 3g-3+n}} C_\alpha L^{2\alpha},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ ranges over $(\mathbb{Z}_{\geq 0})^n$,

$|\alpha| = \sum \alpha_i$ & $L^{2\alpha} = L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$ &

each $C_\alpha > 0$ lies in $\pi^{6g-6+2n-2|\alpha|}\mathbb{Q}$.

**Rmk:** So: $\underbrace{\text{degree}(V_g(L_1, \dots, L_n)) = 6g-6+2n}$

**Rmk:** Mirzakhani explicitly calculates $C_\alpha$ as:

$$C_\alpha = \frac{2^{\delta_{1g}\delta_{1n}}}{2^{|\alpha|} \alpha! (3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \omega^{3g-3+n-|\alpha|}$$

where:

· $\alpha! = (\alpha_1!) \cdots (\alpha_n)!$, $\omega =$ WP symplectic form

$\overline{\mathcal{M}_{g,n}}$ = "Deligne Mumford compactification of moduli space $\mathcal{M}_{g,n}$

$\psi_i$ = Chern class for the cotangent bund along $i^{th}$ puncture.

($\psi_i$: Chern class of bundle $\mathcal{L}_i \to \mathcal{M}_{g,n}$, where

fiber over $X \in \mathcal{M}_{g,n}$ is cotangent bundle of $X$ at $i^{th}$ marked pt)

– unless we have a lot of time later, we will skip this calculation of $C_\alpha$, cause it is not needed going forward.

## Proof of polynomial nature of $V_g$:

proof by induction using recursion formula.

Claim holds for $V_0(L_1, L_2, L_3)$ & $V_1(L_1)$ ✓

By induction, $V_g(L)$ satisfies:

$$\frac{\partial}{\partial L_1} L_1 V_g(L) = A_g^{con}(L) + A_g^{dis}(L) + B_g(L),$$

each is an integral involving smaller complexity volumes, may assume they are polys

$V_g(L)$ is poly of specified form iff $\frac{\partial}{\partial L_1} L_1 V_g(L)$ is.

using induction hypothesis, comes down to calculating integrals:

$$\int_0^\infty x^{2j+1} H(x, t)\, dx$$ ← deg $j+1$ in $t^2$

$$\int_0^\infty \int_0^\infty x^{2j+1} y^{2k+1} H(x+y, t)\, dx\, dy$$ ← deg $j+k+1$ in $t^2$

Lem (calculation) each integral is a polynomial in $t^2$ with each coeff a product of factorials & zeta (non-negative even integer)

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