IV - Generalized McShane Identity

This completes the requisite background. Now we commence in earnest with the work in Mirzakhani's thesis.

The point of departure is:

Thm (McShane 1998) Let X be a hyperbolic once punctured torus. Then
$$\sum_{\substack{\gamma \text{ a scc} \\ \text{on } X}} \frac{1}{1+e^{l_X(\gamma)}} = 1/2.$$

- Amazing: say the fnct $X \mapsto \sum_{\gamma} \frac{1}{1+e^{l_X(\gamma)}}$ is constant on $\mathcal{T}(S_{1,1})$. And we know $\mathcal{T}(S_{1,1}) \cong \mathbb{R}^2$ is pretty big!

Lets use McShane to calculate the volume of moduli space $\mathcal{M}(S_{1,1})$:

Shorthand notation: $\mathcal{T} = \mathcal{T}(S_{1,1})$, $\mathcal{M}$, Mod, ...

set $\mathcal{M}^* = \{(X,\gamma) \mid X \in \mathcal{M},\ \gamma \text{ a scc on } X\}$

$$\pi : \mathcal{M}^* \to \mathcal{M}, \quad \pi(X,\gamma) = X$$

$$L : \mathcal{M}^* \to \mathbb{R}_+, \quad L(X,\gamma) = l_X(\gamma)$$

Set $f(y) = \frac{1}{1+e^y}$. Then McShane is:

$$\forall X \in \mathcal{M}, \quad \sum_{\pi(Y)=X} f(L(Y)) = 1/2$$

fix a scc α on $S_{1,1}$,

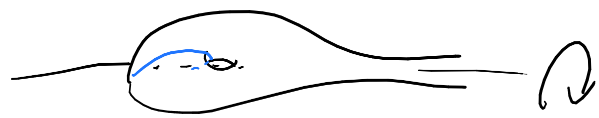
set $Stab(\alpha) := \{ \phi \in Mod \mid \phi(\alpha) = \alpha \}$

Note: Mod acts transitively on set of sccs on $S_{1,1}$

Hence, identify $\boxed{\mathcal{M}^* = \mathcal{T}/Stab(\alpha)}$

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{M}^* = \mathcal{T}/Stab \\ & \searrow & \swarrow \\ & \mathcal{M} = \mathcal{T}/Mod & \end{array}$$

Stab (α) gen by D_α Dehn twist & involution



i acts trivially on Teich.

In FN coords wrt pants decomp α,

have $D_\alpha(l, \tau) = (l, \tau + l)$.

find domain for action of Stab(α)

Hence $\{ (l, \tau) \mid l > 0, \; 0 \le \tau \le l \} /_{(x,0) \sim (x,x)} \cong \mathcal{M}^*$

In coords, if $\mathcal{M}^* \ni Y = (l, \tau)$, then $f(L(Y)) = f(l)$.

WP volume form on $\mathcal{T}$ descends to $dl \wedge d\tau$ on $\mathcal{M}^*$.

Integrate McShane:

$$Vol(\mathcal{M}) = 2 \int_{\mathcal{M}} \sum_{\pi(Y)} f(L(Y)) \, dx = 2 \int_{\mathcal{M}^*} f(L(Y)) \, dY$$

$$= 2 \int_{l=0}^{\infty} \int_{\tau=0}^{l} f(l) \, d\tau \, dl = 2 \int_0^\infty l f(l) \, dl$$

$$= 2 \int_0^\infty \frac{l}{1+e^l} \, dl = \boxed{\pi^2/6}$$

This illustrates Mirzakhani's Strategy for volume recursion:

1) Generalize McShane's Identity for bordered surfaces

2) do clever integrals of const functions over moduli space, breaking things up recursively.

Philosophy: Identity serves as "partition of unity" for action of mapping class group

$\rightsquigarrow$ allows one to reduce action of Mod to action of smaller mapping class groups.

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Generalized McShane–Mirzakhani Identity for Bordered hyp surfaces

(following Wolpert PCMI lecture notes, which in turn follow Tan-Wong-Zhang, "Generalizations of McShane's Id to hyp Surf's" JDG 2006)

Let X be hyperbolic Surf with geodesic boundary, say with no punctures $\partial X = \beta_1, \dots, \beta_n$

Set $\mathcal{A}$ = set of nontrivial free homotopy classes of simple arcs from joining border, homotopy rel border.

= set of nontriv. htpy classes of inj. maps $([0,1], \{0,1\}) \to (X, \partial X)$.

Each elt of $\mathcal{A}$ contains a unique shortest arc, which is a simple geodesic arc orthogonal to boundary.

call these ortho boundary arcs.

Have bijection: $\mathcal{A} \longleftrightarrow$ set of ortho boundary arcs.

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Let $\mathcal{P}$ = set of hyperbolic pants in X with some boundary components also boundary comps of X.

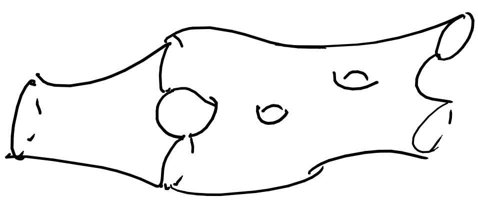
That is, $\mathcal{P}$ = set of all subsurfs $C \subset X$ of form

$$C = \text{component of } X \setminus \begin{pmatrix} \text{maximal set of disjoint simple closed} \\ \text{geodesics in } X \end{pmatrix}$$

with $C \cap \partial X \neq \emptyset$.

2 possibilities for elts of $\mathcal{P}$:

1 - bdry of X



2 bdry comps of X

Fix a boundary compt β of X

Let A_β = elts of A with both endpts on β

$\mathcal{P}_\beta$ = elts of $\mathcal{P}$ with β one of its boundary components
"β-cuff pants"

Claim: Natural bijection $A_\beta \leftrightarrow \mathcal{P}_\beta$

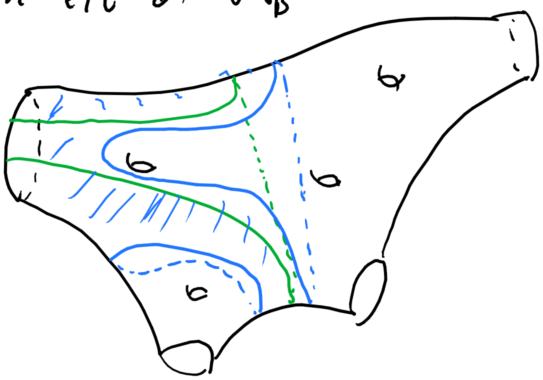
pf: given elt $[\gamma]$ of A_β, choose rep. arc γ, connecting β to β

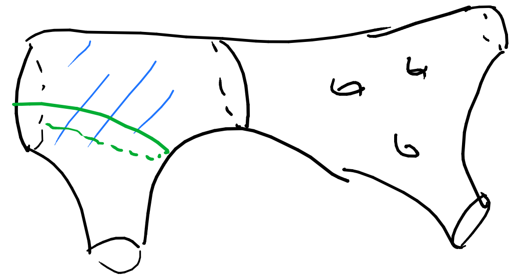
a nbhd N of $\gamma \cup \beta$ is a topological pair of pants: one bdry comp is β, & other 2 are non null homotopic sccs in X

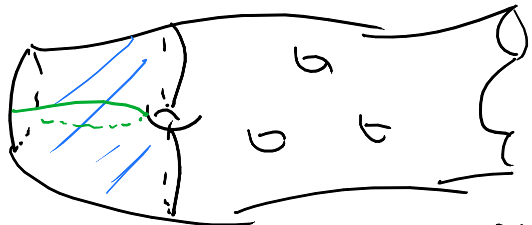
Pull tight to geodesics $\rightsquigarrow$ a geom pants in $\mathcal{P}_\beta$

Clear that does not depend on rep. arc we chose.

Conversely: given $P \in \mathcal{P}_\beta$, there exists a unique htpy class of simple arcs in P connecting β to itself. This gives an elt of $\mathcal{A}_\beta$



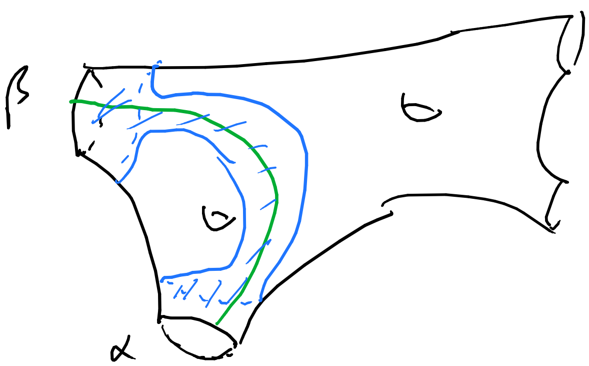


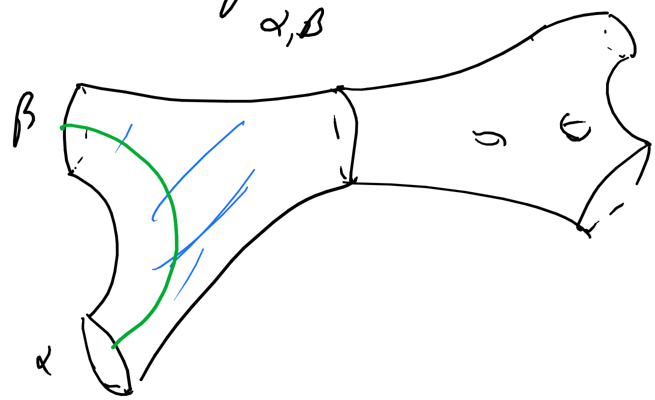


Similarly: If α, β are distinct components of X,

have $\mathcal{A}_{\alpha,\beta}$ = elts $\mathcal{A}$ joining α & β, $\mathcal{P}_{\alpha,\beta}$ = elts $\mathcal{P}$ containing α & β.

Natural bijection $\mathcal{A}_{\alpha,\beta} \longleftrightarrow \mathcal{P}_{\alpha,\beta}$





Now: fix a boundary component β of X

each $x \in \beta$ determines

γ_x: the complete (i.e., maximal) geodesic on X emanating orthogonally from β at x

"ortho-emanating ray"

There are 3 types of ortho emanating ray γ_x:

1) non-simple
2) simple & finite length, terminating at a boundary component of X
   ↳ determines an elt of $\mathcal{A}$.
   - other endpoint may be orthogonal (thus γ_x is an ortho boundary arc)
   or oblique.
3) simple & infinite length.

Each ortho emanating ray γ_x of type (1) or (2) determines a unique elt of $\mathcal{P}$ as follows:

- If γ_x simple & terminates at ∂X, it defines an elt of $\mathcal{A}$ ⇝ gives pants via correspondence above (take nbhd of $\gamma_x \cup$ the 1 or 2 boundary comps)
- If γ_x non-simple, claim that first self intersection pt of γ_x contained in a unique β-cuff pants:

Consider "lasso" subarc beginning at x & ending where γ_x intersects itself first time.

Thicken lasso: boundary of this is a scc & an elt $\alpha \in \mathcal{A}_\beta$

Then geom pants determined by α contains lasso & the int point.

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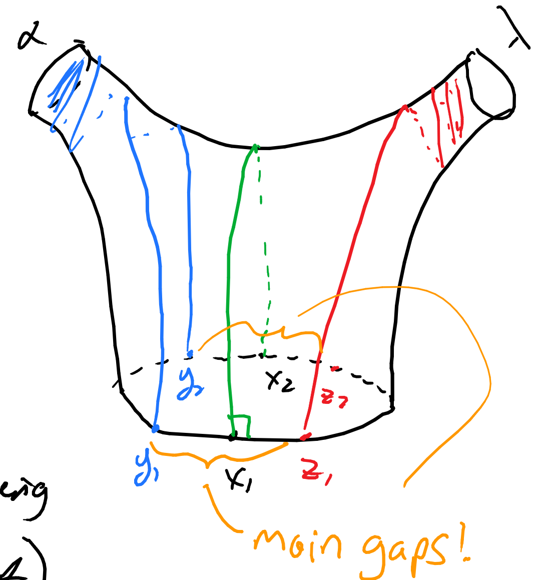
So: each ortho emanating ray γ_x that — simple + finite length $\}$ $\oplus$
— nonsimple $\}$
determines a pants! Further: these are both open conditions in x, +
↳ $P(\gamma_x)$ the pants determined is locally constant

Lets reverse this association:

Fix a β-cuff pants $P \in \mathcal{P}$. Let α, λ be other bdry comps P
Let $\gamma \in \mathcal{A}$ be the unique orthoboundary geod arc in P connecting β to itself. Endpts $x_1, x_2 \in \beta$ (so $\gamma_{x_2} = \gamma = \gamma_{x_1}$)

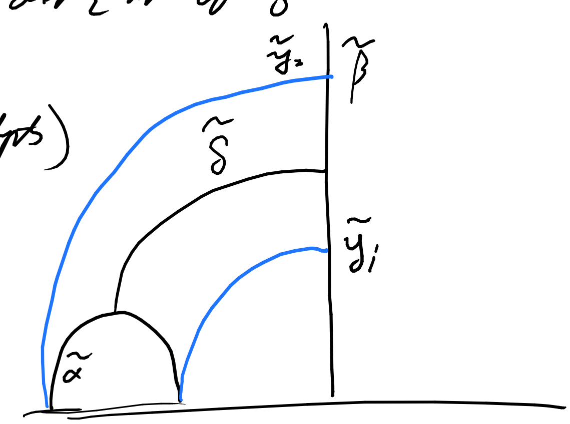
Say that a *spiral* is an infinite simple geod ray that accumulates on a simple closed geodesic.



In P, $\exists$ 2 pts $y_1, y_2 \in \beta$ s.t. the ortho emanating rays γ_{y_1} + γ_{y_2} are spirals accumulating on α (spiraling in opposite directions)

To see this, go in univ cover: lift β to $\tilde{\beta}$ from 0 to ∞
take δ = shortest geod from β to α, lift to $\tilde{\delta}$ starting on $\tilde{\beta}$
+ take $\tilde{\alpha}$ = lift of α through other end of $\tilde{\delta}$

The 2 geodesics in $\mathbb{H}^2$ orth to $\tilde{\beta}$ + asymptotic to $\tilde{\alpha}$ (going to its 2 endpts) project to the simple ortho emanating rays γ_{y_1} + γ_{y_2}.



Similarly, 2 ortho emanating rays γ_{z_1} + γ_{z_2} spiraling toward λ.

<u>Def</u> The P-<u>main gaps along β</u> are the 2 disjoint open subarcs of β that each contain an endpt of γ in interior & have spiral initial pts as endpts

So, in picture main gaps are (y_1, z_1) + $(y_2, z_2$

We also define the <u>P-secondary gaps</u> along β to be the interior of the complement of main gaps.

That is: the intervals (y_1, y_2) + (z_1, z_2)

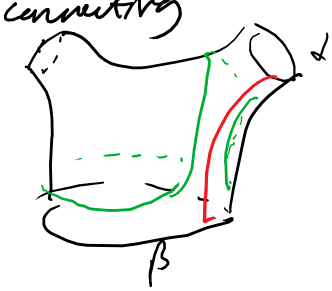
- the shortest arc connecting β to α starts at the midpt of (y_1, y_2), + shortest arc connecting β to λ starts at midpt of (z_1, z_2).

<u>Observe</u>: the ortho-emanating rays starting in P-secondary gaps exit P either through α or λ, and

- the initial segment in P is simple.

(these will be classified / accounted for by behavior elsewhere in surface; in a different pair of pants).

<u>Note</u> If P shares 2 boundary components with X, say $\partial P \cap \partial X = \alpha \cup \beta$. Then for all $w \in$ corresp secondary gap (y_1, y_2) ortho emanating ray γ_w is a simple arc connecting β to α: Thus $\gamma_w \in \mathcal{A}_{\beta, w}$, and P = the pants associated to γ_w



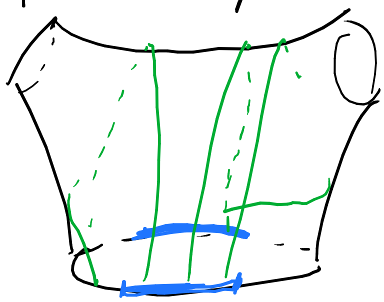
$w \in$ secondary gap $\Rightarrow \gamma_w \in \mathcal{D}$ + $P(\gamma_w) = P$

Key Observation:

Let $w \in$ a P-main gap along β. Then

γ_w is either simple finite length or non simple,

P = the pants determined by γ_w



Notation: orthoemanating ray γ_x.

$\mathcal{D} = \{\gamma_x \mid \text{non simple or simple + finite}\}$

$\mathcal{S} = \{\gamma_x \mid \text{simple infinite}\}$

Thus:

$$\{w \in \beta \mid \gamma_w \text{ determines the pants } P\} = \text{The } P\text{-main gaps along } \beta \left(\cup\ P\text{-secondary gap(s) corresp. to } (\partial P \cap \partial X) \setminus \beta\right)$$

$$\| $$

$$\{w \in \beta \mid \gamma_w \in \mathcal{D} + P(\gamma_w) = P\}$$

Thus we have a corresp between main gaps + orthoemanating rays that are not infinite simple.

Classification of orthoemanating rays:

Thm Let β be a boundary component of X. Then

$$\beta = \underbrace{\{x \mid \gamma_x \text{ simple + orth to } \partial X\}}_{\substack{\| \\ \{\gamma_x \text{ simple infinite}\} \,\cup\, \text{ortho boundary geodesics}}} \cup \{x \mid \gamma_x \text{ nonsimple}\} \cup \{x \mid \gamma_x \text{ simple + boundary oblique}\}$$

In fact, $\{x \mid \gamma_x$ infinite simple$\}$ $\gamma_x \in \mathcal{S}$ is a Cantor set in β

↳ We do not need this, but can kind of see why:

obtained by removing $\underbrace{\rho\text{-main gaps (+ some secondary gaps)}}_{= \text{open intervals}}$

for all β-cuff points;

But we do need:

Thm The set $\{x \mid \gamma_x$ simple $\& \perp \partial X\} = \{x \mid \gamma_x$ simple inf$\} \cup$ ortho boundary geods

has (Lebesgue) measure zero in β.

Proof: We use the following fact

Thm (Birman–Series '85) Let Y be a closed hyp surf.

The union of all complete simple geodesics on Y has Hausdorff dimension 1. Hence measure zero.

go to univ cover & take ε-nbhd of this set

⇝ countable union of thin corridors. Each corresponds to a biinfinite word in fund. group. By analyzing number & width of corridors, they show Haus dim $= 0$

Now: consider our surf X.

Let E = union all simple complete geods $\perp \partial X$

Claim is $E \cap \beta$ has measure 0 in β

Let $\hat{X}$ = double of X along boundary.

Then $\hat{E}$ = double of $E \subset$ complete simple geods on $\hat{X}$

$\Rightarrow$ Haus. dim $\hat{E} = 1$

For small r, If U = r-nbhd $\beta \subset \hat{E}$, Then U is a small hyp annulus & each comp of $\hat{E} \cap U$ = geod are going straight across U.

geom U $\Rightarrow$ $0 = \text{meas}(\hat{E} \cap U) = \sinh r \times \text{Leb meas}(E \cap \beta)$

$\Rightarrow$ Lebesgue measure $E \cap \beta = 0$ ∎

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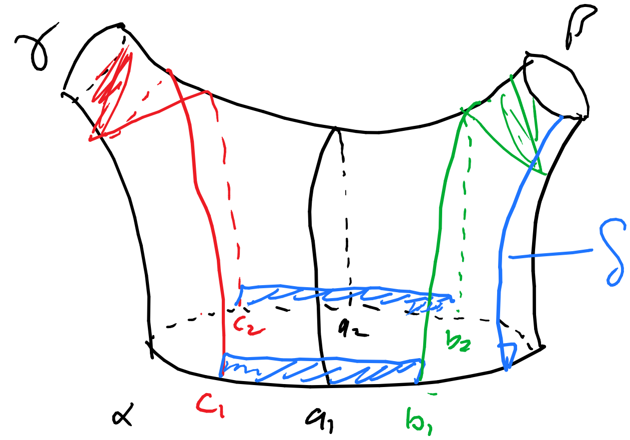
Let $C(x,y,z)$ = unique hyp pants with boundary α, β, γ, of lengths x, y, z

- also allow degenerate case when one or more $x_i = 0$.



Set

$R(x,y,z) = \text{length}(\alpha \setminus (c_1, c_2))$

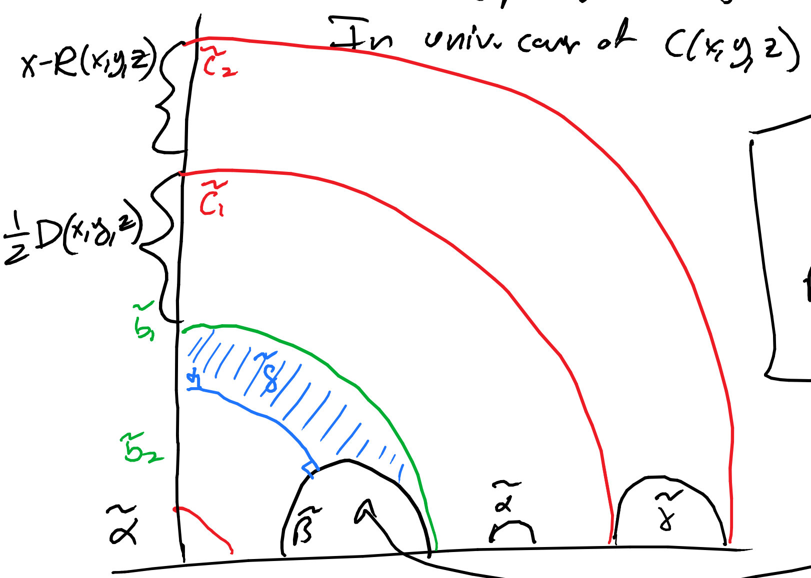
= length main gaps along + Secondary gap assoc β

So $x - R(x,y,z) = $ len (sec. gap assoc γ)

$D(x,y,z) = \text{length}(c_1, b_1) + \text{length}(c_2, b_2)$

= Sum of main gap lengths along α

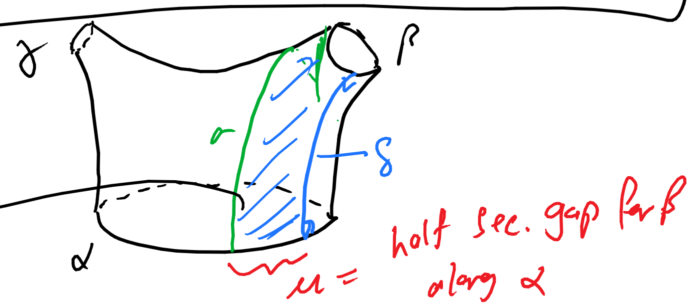
In univ. cover of $C(x,y,z)$



Properties

$$D(x,y,z) = D(x,z,y)$$

$$R(x,y,z) + R(x,z,y) = x + D(x,y,z)$$



geodesic quadrilateral w/ angles $\pi/2, \pi/2, \pi/2, 0$,

hyp trig $\Rightarrow \tanh(\ell(u)) = \operatorname{sech}(\ell(\delta))$

geom of pants $C(x,y,z) \Rightarrow \operatorname{sech}(\ell(\delta)) = \dfrac{\sinh\left(\frac{x}{2}\right)\sinh\left(\frac{y}{2}\right)}{\cosh\left(\frac{z}{2}\right) + \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{y}{2}\right)}$

$2\ell(u) = x - R(x,z,y)$

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hyp trig: $\cosh(\theta + \varphi) = \cosh(\theta)\cosh(\varphi) + \sinh(\theta) + \sinh(\varphi)$

$\tau = \tanh(\theta) = \dfrac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \Rightarrow e^{2\theta} = \dfrac{1+\tau}{1-\tau} \Rightarrow \operatorname{arctanh}(\tau) = \theta = \frac{1}{2}\log\left(\dfrac{1+\tau}{1-\tau}\right)$

So:

$$R(x,z,y) = x - 2\ell(\gamma) = x - 2\,\mathrm{arctanh}\left(\mathrm{sech}\left(\ell(\delta)\right)\right)$$

$$= x - \log\left(\frac{\cosh(\tfrac{z}{2}) + \cosh(\tfrac{x}{2})\cosh(\tfrac{y}{2}) + \sinh(\tfrac{x}{2})\sinh(\tfrac{y}{2})}{\cosh(\tfrac{z}{2}) + \cosh(\tfrac{x}{2})\cosh(\tfrac{y}{2}) - \sinh(\tfrac{x}{2})\sinh(\tfrac{y}{2})}\right)$$

$$x - \log\left(\frac{\cosh(\tfrac{z}{2}) + \cosh(\tfrac{x+y}{2})}{\cosh(\tfrac{z}{2}) + \cosh(\tfrac{x-y}{2})}\right)$$

$$R(x,y,z) = x - 1 \quad \frac{\cosh(\tfrac{y}{2}) + \cosh(\tfrac{x+z}{2})}{\cosh(\tfrac{y}{2}) + \cosh(\tfrac{x-z}{2})}\Bigg)$$

$$D(x,y,z) = R(x,y,z) + R(x,z,y) - x$$
$$= \dots \text{ (exercise!)}$$

$$\boxed{D(x,y,z) = 2\log\left(\frac{e^{x/2} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right)}$$

Also define $H(x,y) = \dfrac{1}{1+e^{\frac{x+y}{2}}} + \dfrac{1}{1+e^{\frac{x-y}{2}}}$

Relations:

- $D(0,0,0) = 0$
- $R(0,0,0) = 0$
- $\dfrac{\partial}{\partial x} D(x,y,z) = 2\left(\dfrac{\frac{1}{2}e^{x/2}}{e^{x/2} + e^{\frac{y+z}{2}}} - \dfrac{-\frac{1}{2}e^{-x/2}}{e^{-x/2} + e^{\frac{y+z}{2}}}\right) = H(y+z, x)$
- $2\dfrac{\partial}{\partial x} R(x,y,z) = \dots = H(z, x+y) + H(z, x-y)$

Putting this all together:

Given hyp surf X w/ geodesic boundary ∂X.

- $\mathcal{P}$ set of hyperbolic boundary cuff pants.
- each $P \in \mathcal{P}$ has main gaps + secondary gaps along boundary
- $x \in \partial X \rightsquigarrow \gamma_x$ ortho emanating ray.

  $\gamma_x \in \mathcal{S}$ if simple + infinite

  $\gamma_x \in \mathcal{D}$ if non simple or simple finite.

  each $\gamma_x \in \mathcal{D}$ determines a pants $P(\gamma_x)$ w/ $x \in \partial P$,

Key observations:

☆ If $x \in \beta$ component of ∂X, $P_0 \in \mathcal{P}$ with $\beta \subset \partial P_0$

$x \in \mathcal{D}$ with $P(\gamma_x) = P_0 \iff$ $x \in P_0$-main gap along β

or

$x \in P_0$-secondary gap along β

assoc. to $(\partial P_0 \cap \partial X) \setminus \beta$

☆ Set $\{x \mid \gamma_x \in \mathcal{S}\}$ has measure zero in ∂X

Thm (Mirzakhani)

Let β be a boundary component of hyp surf X. Then

$$\text{Length}_X(\beta) = \sum_{\substack{P\ \beta\text{-cuff} \\ \text{hyp pants}}} \text{len}\begin{pmatrix}P\text{-main gaps} \\ \text{along } \beta\end{pmatrix} + \sum_{\substack{\alpha \subset \partial X \\ \alpha \neq \beta}} \sum_{\substack{P\ \alpha\text{-cuff} \\ + \beta\text{-cuff} \\ \text{hyp pants}}} \text{length}\begin{pmatrix}P\text{-secondary gap along} \\ \beta \text{ corresp to } \alpha\end{pmatrix}$$

using our formulas for the gap lengths, get:

McShane - Mirzakhani Identity

Let X be hyp surf genus g, with n boundary components $\beta_1, \dots, \beta_n$ of length $L_1, \dots, L_n$. Then

$$L_1 = \sum_{(\alpha_1, \alpha_2) \in \mathcal{J}_1} D(L_1, \ell_X(\alpha_1), \ell_X(\alpha_2)) + \sum_{i=2}^{n} \sum_{\gamma \in \mathcal{J}_{1,i}} R(L_1, L_i, \ell_X(\gamma))$$

where

$\mathcal{J}_i$ = set of unordered pairs (α_1, α_2) of essential simple closed curves bounding pants with β_i s.t. α_1, α_2 not $\sim$ boundary

$i \neq j$: $\mathcal{J}_{i,j}$ = set of essential curves γ not $\sim$ boundary s.t. β_i, β_j, γ bound a pants

Notice; as $x \to 0$

$$D(x,y,z) \sim x\, H(y+z, x) \sim \frac{2x}{1+e^{\frac{y+z}{2}}}$$

$$R(x,y,z) \sim x\, H(z,y), \qquad R(x,x,z) \sim \frac{2x}{1+e^{z/2}}$$

as $L_1 \to 0$, both sides in Id $\to 0$, divide by $L_1 \rightsquigarrow$

Cor X hyp surf as above, boundary $\beta_1, \dots, \beta_n$ of lengths $L_1 = 0$ & $L_2, \dots, L_n \geq 0$
assume $L_1 = 0$. Then a puncture.

$$\frac{1}{2} = \sum_{(\alpha_1, \alpha_2) \in \mathcal{J}_1} \frac{1}{1+e^{\frac{\ell_X(\alpha_1) + \ell_X(\alpha_2)}{2}}} + \sum_{i=2}^{n} \sum_{\gamma \in \mathcal{J}_{1,i}} \left(\frac{1}{1+e^{\frac{\ell_X(\gamma) + L_i}{2}}} + \frac{1}{1+e^{\frac{\ell_X(\gamma) - L_i}{2}}} \right)$$

$\Longrightarrow$ McShane Id when $n=1$.

(Then $\mathcal{J}_{1,i} = \emptyset$ & $\mathcal{J}_1 = \{ (\gamma, \gamma) \mid \gamma \text{ a scc on } X \}$)

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