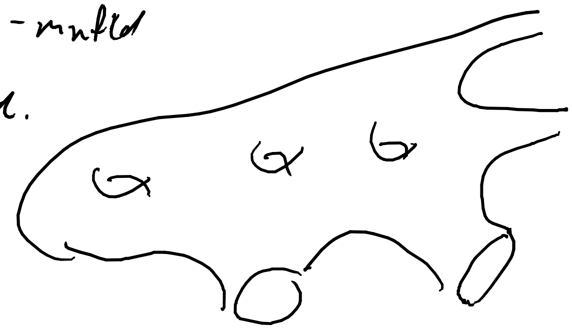
III Teichmüller Space, Mapping Class Group & Moduli Space

For us, a surface is an orientable smooth 2-manifold.

Sometimes we allow boundary, sometimes not.



Let S_g = closed surface of genus g.

$S_{g,n}^b = S_g$ minus $b \geq 0$ disjoint open discs, and
$n \geq 0$ disjoint points.

(so $S_g = S_{g,0}^0$)

— surface of genus g with n punctures & b boundary components

Euler Characteristic $\boxed{\chi(S^b_{g,n}) = 2-2g-b-n}$ do MCG!

Classification of surfaces: Every finite-type surface is homeomorphic to $S^b_{g,n}$ for some unique $g, b, n \geq 0$.

Fix a surface S. The mapping class group of S is

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S) / \mathrm{Homeo}_0(S), \text{ where}$$

$\mathrm{Homeo}^+(S)$ = group of orientation preserving homeos S

$\mathrm{Homeo}_0(S)$ = connected component of Id_S ← Isotopy

$$= \left\{ f \in \mathrm{Homeo}^+(S) \;\middle|\; \begin{array}{l} \exists\, h: S \times [0,1] \to S \text{ s.t.} \\ h(\cdot,0) = Id,\ h(\cdot,1) = f,\ h(\cdot,t) \in \mathrm{Homeo}\ \forall t \end{array} \right\}$$

normal subgroup!

Homeomorphisms f, g are <u>isotopic</u> if they lie in the same $\mathrm{Homeo}_0(S)$ coset that is: iff $\exists\, h : S \times [0,1] \to S$ cont s.t.
$h_0 = f$, $h_1 = g$ h_t homeo $\forall t$, $h_t(x) = h(x,t)$.

<u>Thm</u> (Hamstrom 60's) $\mathrm{Homeo}_0(S_{g,n}^b)$ is contractible, provided

$$S_{g,n}^b \notin \{ S_2, S_1, \underset{\mathbb{R}^2}{\underset{\shortparallel}{S_{0,1}^0}}, \underset{\text{disk}}{\underset{\shortparallel}{S_{0,0}^1}} \}$$

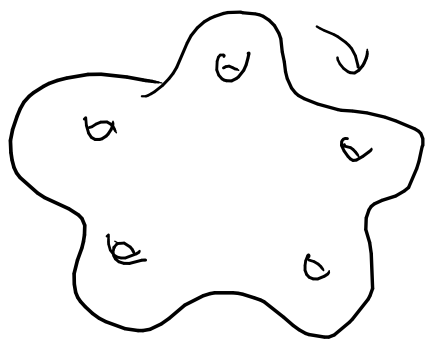
<u>Facts:</u> $\mathrm{Homeo}^+(S)$ is a huge (uncountable, ∞-dim'l etc) top. group (e.g w/ compact open topology)

Homeo_0 = closed normal subgrp; path / connected comp. of Id, contractible

quotient $\mathrm{Mod}(S)$ inherits quotient topology; & this is <u>discrete topology</u>

In fact: $\mathrm{Mod}(S)$ is finitely generated group!
(though this is not so necessary for us)

<u>Ex</u> · finite order elements

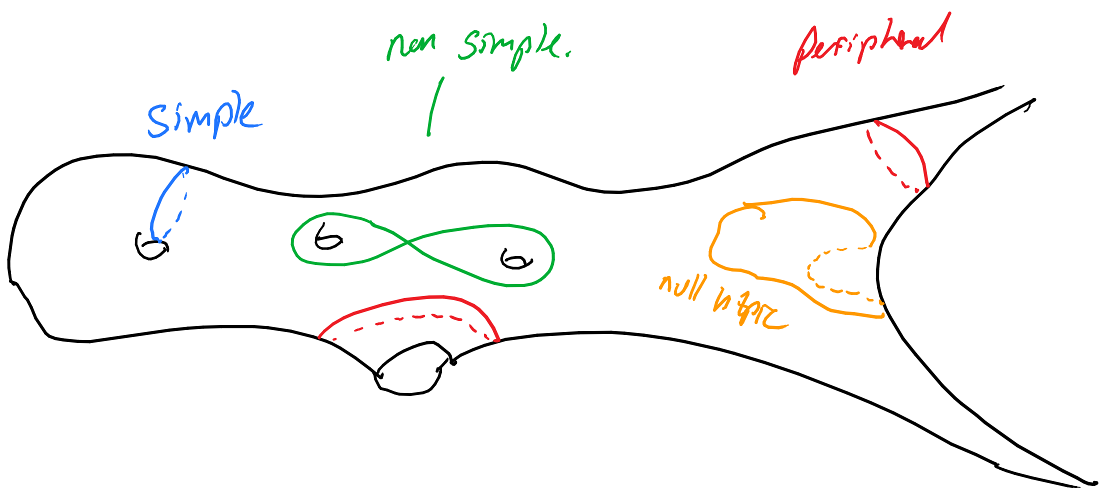


<u>Defn</u> a (homotopy class of) curve on S is

· <u>simple</u> if its htpy class contains an embedding $S^1 \to S$

· <u>peripheral</u> if not htpic into every nbhd of a puncture / boundary component

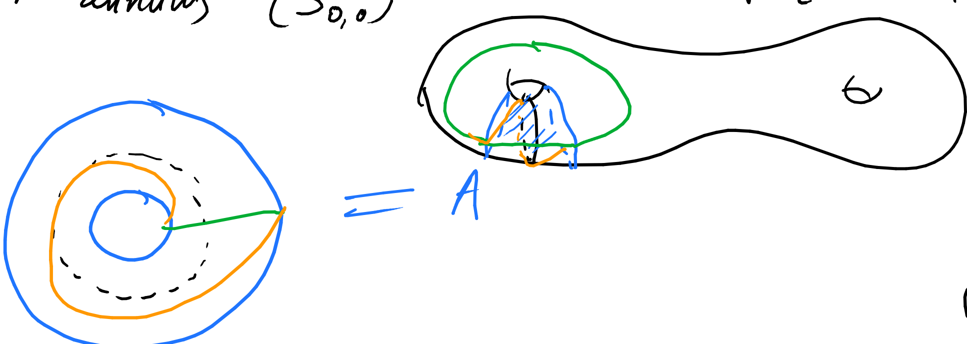
<u>essential</u> if neither peripheral nor null homotopic.



· <u>Dehn Twists</u> Let α be a (htpy class of) essential simple closed curve on S.

may take a nbhd of α whose closure $A \subset S$ is topologically an annulus ($S_{0,0}^{2}$). Parametrize $A = \{z \in \mathbb{C} \mid 1 \le |z| \le 2\}$

└ preserving orientation!



$re^{i\theta}e^{2\pi i(r-1)}$

define homeo $h: S \to S$

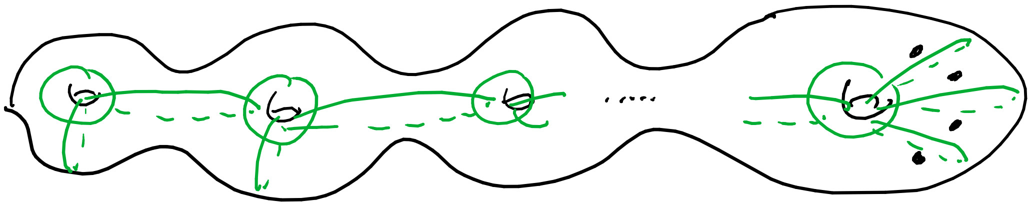
$$h_A(x) = \begin{cases} x & x \notin A \\ re^{i\theta}e^{2\pi i(r-1)}, & x = re^{i\theta} \in A \cong \{z \in \mathbb{C} \mid 1 \le |z| \le 2\} \end{cases}$$

<u>Defn.</u> The <u>Dehn Twist about α</u> is

D_α = isotopy class of $h_A \in Mod(S)$

Fact: Mod(S) is finitely generated by Dehn twists.

In fact $Mod(S_{g,n})$ gen by twists about curves:



Cusps / punctures / boundary components for hyp surfaces:

Before, we defined hyp surf to be $\mathbb{H}^2/\Gamma$, some $\Gamma \leq PSL(2,\mathbb{R})$ acting free prop disc.

In this case $\mathbb{H}^2/\Gamma$ is always a top surface without boundary.

- So will be homeo to $S_{g,n}$ some g,n

Need to discuss what kind of geometry we permit near punctures

- Also: we will want hyp surfaces homeo to S_g^b, $b \geq 1$, so must expand our defn.

Say $\Gamma \leq PSL(2,\mathbb{R})$ act freely prop disc on $\mathbb{H}^2$.

set $X = \mathbb{H}^2/\Gamma$. Then X 2-dim surf with $\pi_1(X) \cong \Gamma$

so X has finite type iff Γ f-gen group.

In this case $X \cong S_{g,n}$ some $g,n \geq 0$

each puncture of $X \Longleftrightarrow$ well-def conj class in $\pi_1(X)$

$\Longleftrightarrow$ conj class in

punctures are "ends" of X: complements of cpct sets

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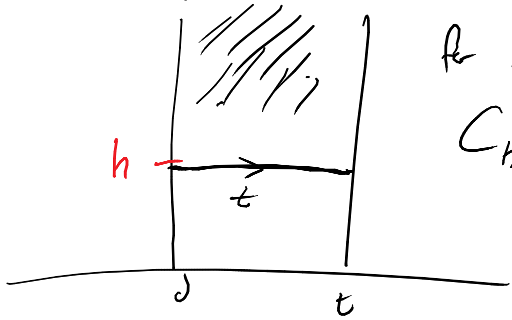
Say an end of X is

- a **cusp** if corresp conj class in Γ is parabolic
- **flaring** if ______, ______ hyperbolic.

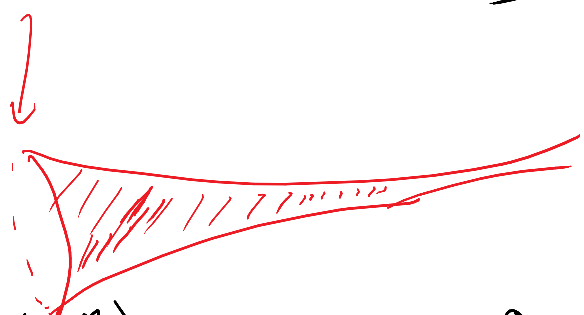
Fix end e of X & choose $\varphi \in \Gamma$ in corresp conj class

Case 1 φ parabolic; assume $\varphi = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ some t;



for h large, consider

$$C_h = \left\{ z \in \mathbb{H}^2 \;\middle|\; \begin{array}{l} 0 \le \mathrm{Re}(z) < t \\ \mathrm{Im}(z) \geq h \end{array} \right\}$$



for suff large h,

C_h embeds into X
(since end is punctured disc, $\pi_1 = \mathbb{Z}$, gen by φ)

$$\text{Area } C_h = \int_{x=0}^{t} \int_{y=h}^{\infty} \frac{1}{y^2}\, dx\, dy = \int_0^t \left[\frac{-1}{y}\right]_h^{\infty} dx$$

$$= \int_0^t \frac{1}{h}\, dx = \frac{t}{h} < \infty$$

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So in this case the cusp has finite area & is modeled on

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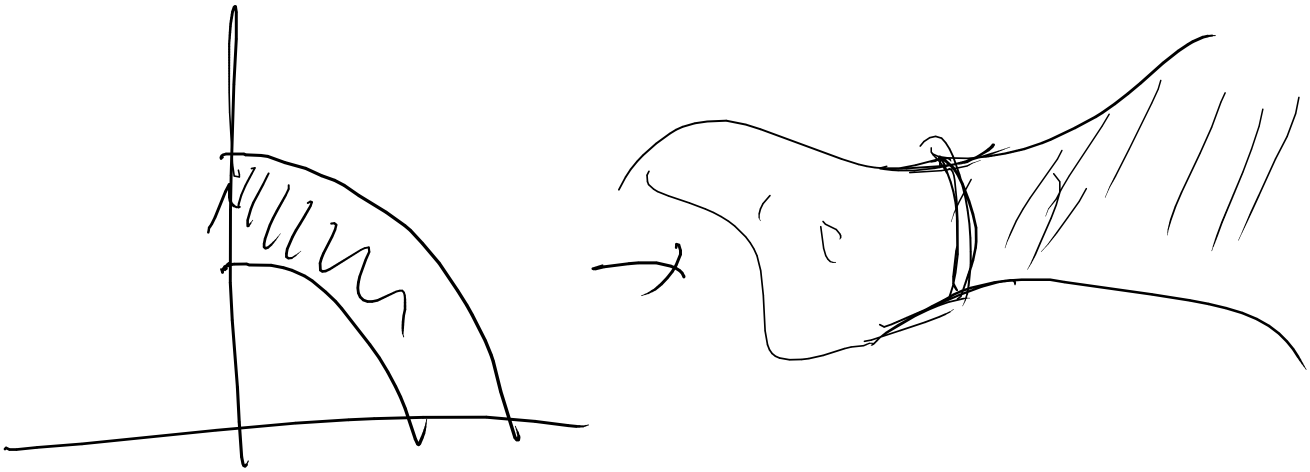
Case 2 φ hyperbolic.

This embeds into X has infinite area!

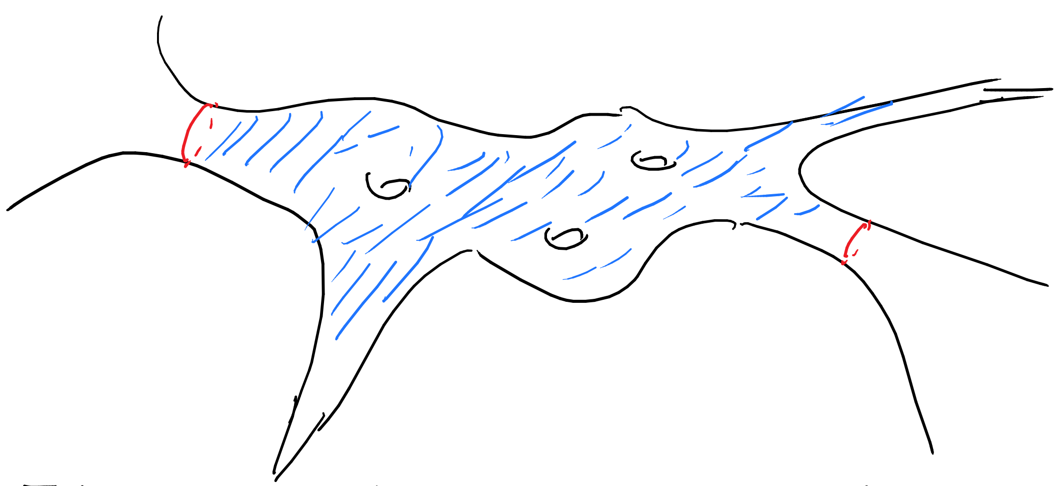


<u>Thus</u> if $X = \mathbb{H}^2/\Gamma$ has finite type,

then X has finite area iff each end is a cusp,

inf area if some end flares.

For each flaring end, $\exists!$ closed geodesic on X rep. that conj class. If we <u>cut on that geodesic</u> result is a hyperbolic surface with <u>geodesic boundary</u> + finite area



—This is our defn of a hyperbolic surface with boundary:

$\mathbb{H}^2\backslash\Gamma$ with the flaring ends removed.

Henceforth, this is what we mean by <u>finite-area hyperbolic surf</u>

(note: $\cong S_{g,n}^b$) ——— Always has geodesic boundary!

Defn Given a top surface $S_{g,n}^b$, a <u>hyperbolic structure</u> on $S_{g,n}^b$ is a homeomorphism $S_{g,n}^b \xrightarrow{\cong} X$, where X is a finite-area hyp surface.

This data is equivalent to choosing $\rho \in \mathrm{Hom}(\pi_1(S_{g,n}^b), PSL(2,\mathbb{R}))$

s.t.

- $\mathrm{Image}(\rho)$ acts freely & prop disc. on $\mathbb{H}^2$
  ($\iff \rho$ discrete & faithful)
- $\rho(c)$ parabolic for each conj. class c of a puncture of $S_{g,n}^b$
- $\rho(c)$ hyperbolic for each boundary component.

$$\left(\rho \xmapsto{\text{equivalence}} X = \mathbb{H}^2/_{\mathrm{Im}(\rho)} \text{ minus flaring ends} \right)$$

Defn Let $S = S_{g,n}^b$. The <u>Teichmüller Space</u> of S is the space of all homotopy classes of hyperbolic structures on S.

More precisely:

$$\mathcal{T}(S) = \left\{ (X,f) \;\middle|\; \begin{array}{l} X \text{ a hyp. surface (finite area w/ geod. boundary)} \\ \text{and } f: S \to X \text{ a homeomorphism} \end{array} \right\} \Big/ \sim$$

where $(X,f) \sim (Y,g)$ if $g \circ f^{-1}: X \to Y$ is homotopic to an isometry.

$$\begin{tikzcd} & X \arrow[d, dashed, "\sim \text{ isom}"] \\ S \arrow[ur, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

• The homeo $f: S \to X$ is called the "marking" — tells S how to "wear" the geometry of the hyp surface X.

Length Functions

Let $\mathcal{C}(S)$ = set of homotopy classes of curves on S that are
- not nullhomotopic
- not homotopic (into every nbhd of) a puncture

(so the b boundary curves count)

Let $\mathcal{S}(S) \subset \mathcal{C}(S)$ be subset of simple curves.

Each $\alpha \in \mathcal{C}(S)$ gives length function $\ell_\alpha : \mathcal{T}(S) \to \mathbb{R}_+$

given $(X, f) \in \mathcal{T}(S)$, $S^1 \xrightarrow{\alpha} S \xrightarrow{f} X$

$f(\alpha)$ is a curve on hyp surf X, so set

$\ell_\alpha([(X,f)])$ = hyp length of (htpy class of) curve $f(\alpha)$ in X.

Length functions give a map

$$\begin{aligned} \mathcal{T}(S) &\xrightarrow{\Phi} \mathbb{R}_+^{\mathcal{S}(S)} \\ X &\longmapsto \{\ell_\alpha(X)\}_{\alpha \in \mathcal{S}(S)} \end{aligned}$$

Fact (will be more clear later on):

This map is injective! (a hyp metric on S is determined by the lengths of its curves)

Topology on $\mathcal{T}(S)$:

Give $\mathcal{T}(S)$ subspace topology inherited from $\left(\mathbb{R}_+^{\mathcal{S}(S)}, \text{ product topology}\right)$

That is: st Φ is homeo between $\mathcal{T}(S)$ & its image.

(Recall product topology)

Alternately:

<u>Defn</u> We give $\mathcal{T}(S)$ the coarsest topology s.t.

$$\ell_\alpha : \mathcal{T}(S) \to \mathbb{R}_+ \text{ is cont for each } \alpha \in \mathcal{S}(S)$$

Thus: $U \subset \mathcal{T}(S)$ is open iff

$\forall x \in U$, $\forall \alpha_1, \dots, \alpha_n \in \mathcal{S}(S)$, $\exists\ \varepsilon_1, \dots, \varepsilon_n$ s.t.

$$\{ Y \mid \ell_{\alpha_i}(Y) \in [\ell_{\alpha_i}(X) - \varepsilon_i, \ell_{\alpha_i}(X) + \varepsilon_i] \ \forall i = 1, \dots, n \} \subset U$$

<u>2nd Defn of topology:</u>

fix $p \in S$

$$DF(S) = \left\{ \rho \in \mathrm{Hom}(\pi_1(S,p), PSL(2,\mathbb{R})) \;\middle|\; \begin{array}{c} \rho \text{ is discrete, faithful,} \\ \rho(\text{each puncture}) \text{ parabolic } \& \ \rho(\partial) \text{ is hyperbolic} \end{array} \right\}$$

$PSL(2,\mathbb{R})$ is a Lie group, so has top.

Choose generators $\gamma_1, \dots, \gamma_m$ of $\pi_1(S,p)$.

since $\rho \in \mathrm{Hom}(\pi_1(S,p), PSL(2,\mathbb{R}))$ determined by

$\rho(\gamma_1), \dots, \rho(\gamma_m)$ view

$$DF \subset \mathrm{Hom}(\pi_1(S,p), PSL(2,\mathbb{R})) \subset PSL(2,\mathbb{R})^m$$

$\Rightarrow$ give DF subspace topology.

$PSL(2,\mathbb{R})$ acts on $PSL(2,\mathbb{R})^m$ by conj, leaves DF invariant

$\Rightarrow$ give $DF/PSL(2,\mathbb{R})$ quotient topology

$\quad \hookrightarrow = \mathcal{T}(S)$

The mapping class group act on Teich space by changing the marking:

$$\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$$

$$\phi\cdot[(X,f)] = [(X, f\circ\phi^{-1})] \qquad \phi \circlearrowright S \xrightarrow{f} X$$

(if $\phi_1 \sim \phi_2$, then $(X, f\circ\phi_1^{-1}) \sim (X, f\circ\phi_2^{-1})$;

$$\begin{tikzcd} S \arrow[r, bend left, "\phi_1^{-1}"] \arrow[r, bend right, "\phi_2^{-1}"'] & S \arrow[r] & f \end{tikzcd} \qquad f\circ\phi_2^{-1}\circ\phi_1\circ f^{-1} \simeq Id$$

Action is clearly by homeomorphisms.

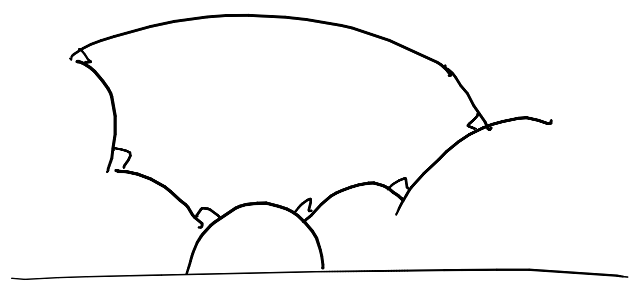
Fact: action $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$ is properly discontinuous
(can't prove yet; maybe later...)

(Next Goal: Coordinates on $\mathcal{T}(S)$)

Hyperbolic Hexagons

Need to study right-angled hyperbolic hexagons in $\mathbb{H}^2$

= hexagon whose edges are geodesic segments meeting at right angles



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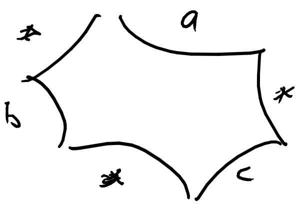
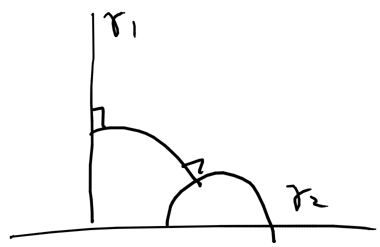
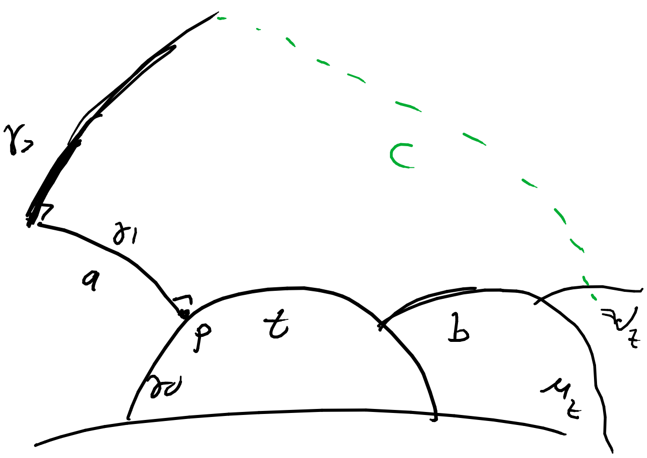
Lemma $\forall a, b, c > \mathbb{R}$, up to isometry $\exists !$ right angled hexagon in $\mathbb{H}^2$ with alternating side lengths $a\ b\ c$ (in ccw order) oriented



proof:

Use fact: for any pair of infinite geods γ_1, γ_2 in $\mathbb{H}^2$ with 4 distinct endpts $\exists !$ int geodesic perp to both of them





fix a geod γ_0 & basept p.

γ_1 : geod $\perp$ to γ_0 at p

γ_2 = geod $\perp$ to γ_1 at dist a from p

for each $t > 0$, let μ_t = geod $\perp$ γ_0 dist t from p

ν_t = geod $\perp$ to μ_t dist b from γ_0

exists t_0 s.t. ν_t & γ_2 share endpt

as $t \to \infty$, dist from ν_t & $\gamma_2 \to \infty$

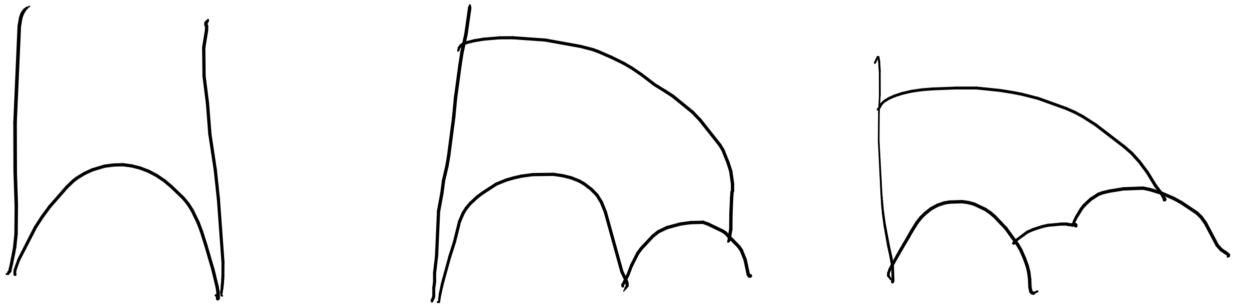
easy to see perp dist γ_2 to ν_t is monotone increasing, cont function,

IVT $\Rightarrow \exists !\ t$ s.t. perp dist γ_2 to ν_t is c.

Thus we've constructed a hyp hexagon,

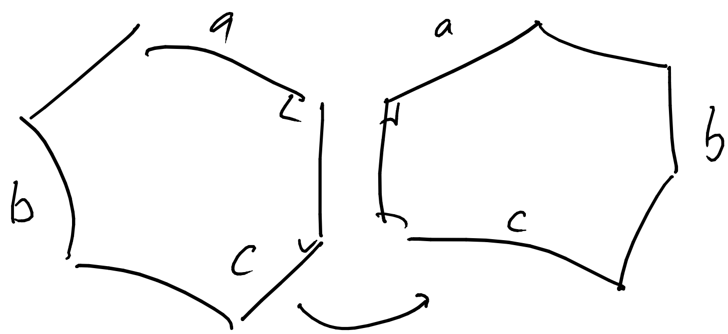
Clear that it is unique up to isometry ∎

Note: also have unique (up to isometry) 'degenerate hexagons where a, b, c allowed to be 0:

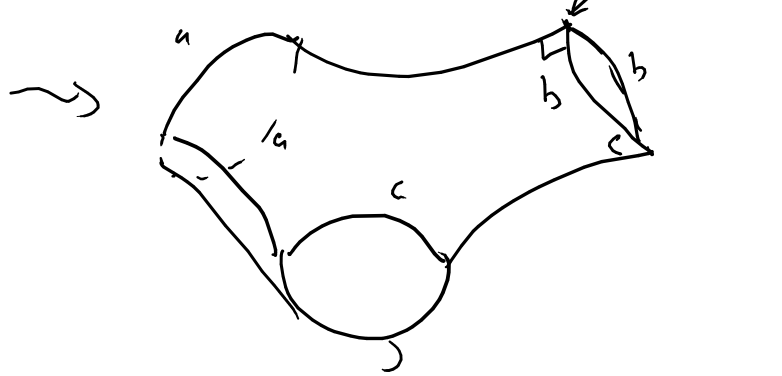


0,0,0 $\qquad$ $\partial 0 +$ $\qquad$ $0++$

Now: given any $a,b,c>0$, Take 2 copies of corresp. hyp hexagon: (with opp. orientation of orderings)



glue along the other 3 sides:



Pair of pants!

result: hyp surf homeomorphic to $S^3_{0,0}$

with geodesic boundary of lengths $2a, 2c, 2b$.

The Teichmüller space of a pair of pants:

Let $P \cong S^3_{0,0}$, with boundary components α, β, γ.

top surface

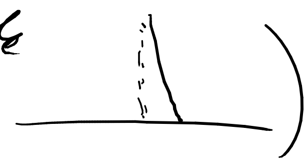
Prop The map $\mathcal{T}(P) \xrightarrow{\Phi} \mathbb{R}_+^3$

$$X \mapsto (\ell_X(\alpha), \ell_X(\beta), \ell_X(\gamma))$$

is a homeomorphism.

proof: First show bijection:

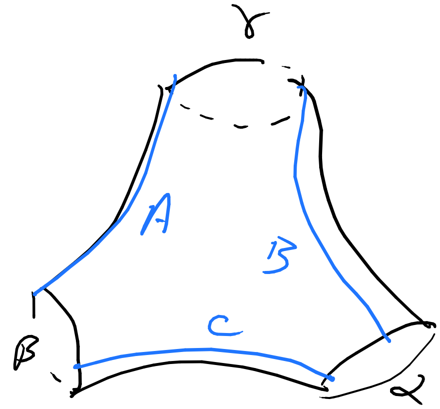
$X = [(X, f)] \in \mathcal{T}(P)$, so X hyp surf w/ geod bdry, $f: P \to X$ homeo. In X, for pair of boundary components, $\exists!$ shortest path connecting them. This will be a geodesic perp to the boundary (variational principle)



Call A = segment connecting $f(\beta), f(\gamma)$

B = ———— " $f(\alpha), f(\gamma)$

C = ———— " $f(\alpha), f(\beta)$



Let H_1, H_2 = closures of the 2 components $X \setminus (A \cup B \cup C)$

Each H_i = right angled hyp hexagon with 3 alternating side length $\ell(A), \ell(B), \ell(C)$

$\Rightarrow H_1$ & H_2 abstractly isometric

$\Rightarrow$ The other 3 sides of H_1 & H_2 have same lengths

So arcs A, B, C must cut each bndry cpt of X in half: so H_1 & H_2 are the unique hyp hexagon with alt side lengths $\frac{l_X(\alpha)}{2}, \frac{l_X(\beta)}{2}, \frac{l_X(\gamma)}{2}$

We now see

<u>Surjectivity:</u> Given $x_1, x_2, x_3 \in \mathbb{R}_+^3$, take 2 hyp hexagons with alt side length $\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}$ (both orientations) glue on other sides to get hyp pair of pants X with bdry length x_1, x_2, x_3, take $P \to X$ the obvious homeo.

<u>Injectivity</u> If X, Y are 2 hyp pairs of pants w/ same bdry lengths, above shows X & Y are isometric (both obtained by gluing isometric hexagons)

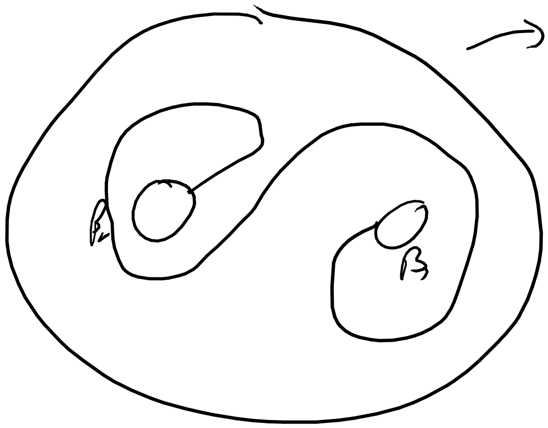
Thus $X = Y$ in $\mathcal{T}(P)$ since $\exists!$ isotopy class of homeos $P \to P$ preserving each boundary component (below).

<u>Now, to see Φ homeo:</u> clear Φ continuous (by def of top on $\mathcal{T}(P)$).

To see Φ^{-1} cont, notice that if 2 hexagons have close side lengths, then the repns $\pi_1(P) \to PSL(2,\mathbb{R})$ that we build (via doubling & gluing) are close in $\text{Hom}(\pi_1(P), PSL(2,\mathbb{R}))$! $\blacksquare$

Facts used in the proof above:

1) In pair of pants P, for any pair of distinct boundary components β_0, β_1
$\exists!$ htpy class of simple arcs $\gamma : [0,1] \to P$ with
$$\gamma(0) \in \beta_0, \quad \gamma(1) \in \beta_1$$

proof: 

→ send 3rd bdy to infinity

Think of being in $\mathbb{R}^2 \setminus 2$ disks

Seems clear that htpic to 

...

real argument: curves must be disjoint up to homotopy
- find innermost bigon, can always get rid of it cause in plane.
- remove bigons, then disjoint.

Now cut on both curves, one component of complement is disk → gives you the homotopy between them. ∎

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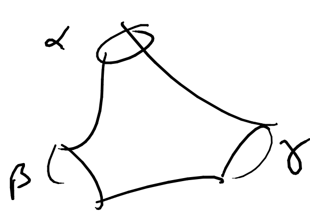
2) Let $PMod(S^3_{0,0}) \leq Mod(S^3_{0,0})$ be subgroup preserving each boundary component. Claim: $PMod(S^3_{0,0}) = \{1\}$

Proof: Given $f \in Homeo^+(S^3_{0,0})$ preserving each bdy cmpt,

$P = S^3_{0,0}$ 

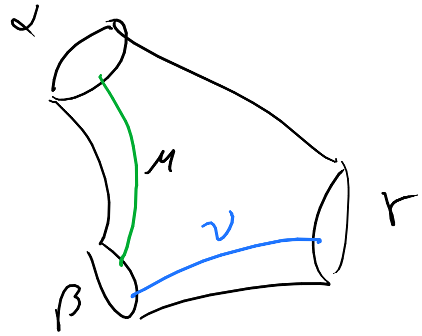
Isotope f to fix α ptwise

Take arc μ from α to β

Then $f(\mu) \sim \mu$

so may adjust f by isotopy st $f|_\mu = Id$

further adjust f s.t. $f|_\beta = Id$



disj μ

Next take ν arc β to γ, again. adjust f by isotopy

st $f|_\nu = Id$ & $f = Id$ on γ.

Now: cut on μ & ν, result:

$$P \setminus \mu \cup \nu = \text{disk} \cong D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

& $f : D \to D$ homeo with $f|_{\partial D} = Id$.

Now Def $F : D^2 \times I \to D^2$

$$t < 1 \qquad F(z,t) = \begin{cases} (1-t) f\left(\frac{z}{1-t}\right) & , 0 \leq |z| \leq 1-t \\ z & , 1-t \leq |z| \leq 1 \end{cases}$$

(Alexander Lemma!)

$t = 1 \qquad F(z,1) = z \qquad$ (identity)

Then F gives isotopy from f to Id ∎

Fenchel-Nielsen Coordinates on $\mathcal{T}(S)$

For simplicity, say $S = S_g$ closed, $g \geq 2$ (mention other cases later

We will build a homeo $\boxed{\mathcal{T}(S_g) \to \mathbb{R}^{6g-6}}$

First choose coordinate system of curves on S_g:

maximal set of disjoint simple closed curves

- a pants decomposition $\gamma = \{\gamma_1, \dots, \gamma_{3g-3}\}$ of disjoint simple closed curves on S_g

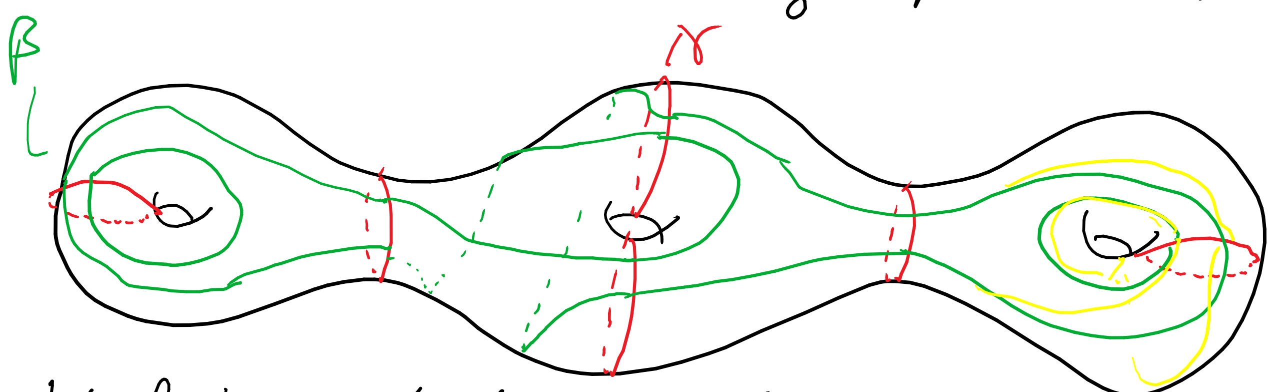
  exercise: any pants decomp has $3g-3$ curves

  ↳ these cut S_g into $2g-2$ pairs of pants $P_1, \dots, P_{2g-2}$

- a set $\beta = \{\beta_1, \dots, \beta_m\}$ of "seams"

  = a set of disjoint sccs s.t. for each P_i,

  $(P_i) \cap (\cup \beta_j)$ = 3 disjoint arcs pairwise connecting the bdry components of P_i.



- lots of choices to build β; just draw 3 arcs in each pants & then match up the endpts however you want
- # curves in β not predetermined!

Using this coordinate system, to each $X \in \mathcal{T}(S_g)$, we associate a <u>length parameter</u> & <u>twist parameter</u> for each γ_i:

- The <u>length parameter</u> of γ_i is $\ell_i(X) = \ell_X(\gamma_i) \in \mathbb{R}^+$

- <u>Twist parameters</u>

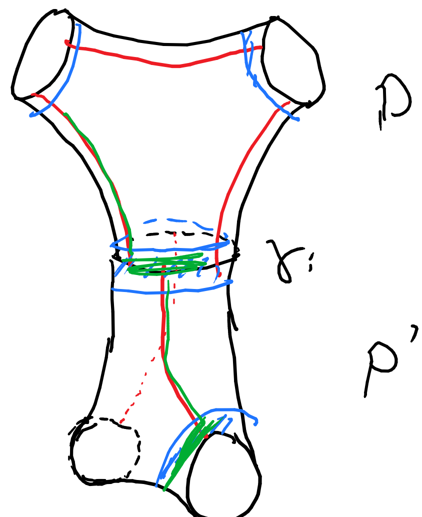
$(X, f) \in \mathcal{T}(S_g)$, X hyp surf, $f: S_g \to X$

realize $f(\gamma_1), \dots, f(\gamma_{3g-3})$ as geodesics in X.

cut X into hyperbolic pants with geod. boundary.

On each pants, draw the 3 geodesic arcs pairwise connecting boundary components.

Call δ = union of all of these



Around each $f(\gamma_i)$, take small metric nbhd.

call η = union of these annuli

each seam curve $f(\beta_j)$ may homotope st it lies in $\boxed{\delta \cup \eta}$

(alt, homotope to lie in $\delta \cup (\bigcup f(\gamma_i))$)

<u>Twist parameter of γ_i:</u> there are 2 arcs in β that cross γ_i (maybe same curve?). In X, after our homotopy, each one comes to $f(\gamma_i)$ along δ, turns left or right & wraps around annulus some, leaves along another arc of δ.

Def twist parameter of γ_i: Choose either seam arc crossing γ_i

$\tau_i(X) =$ total signed $\begin{pmatrix} \text{turn right } + \\ \text{left } - \end{pmatrix}$ horiz displacement along $f(\gamma_i)$ from entering arc of δ to exiting arc of δ $\in \mathbb{R}$

- sign +/− well defined indep of which way you approach!

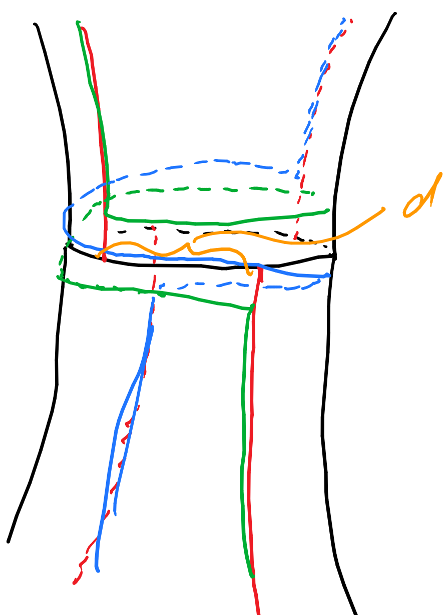
Independent of which seam arc you choose!

If there are 4 arcs of δ incident on $f(\gamma_i)$;

- each seam arc takes 2 of them.
- endpts are diametrically opposite along $f(\gamma_i)$ (divide geod. in half)
- Thus fact that seam arcs are disjoint forces horiz displacement to agree!

green:
$\tau = -\ell_{\gamma_i}(X) - d$

blue:
$\tau = -\ell_{\gamma_i}(X) - d$



Rmk changing seam curves β changes $\tau_i(X)$ by $\frac{k}{2}\ell_i(X)$ some $k \in \mathbb{Z}$

Prop For the given coord system of curves ($\{\gamma_i\}$ pants, $\{\beta_j\}$ seams) the map

$$FN : \mathcal{T}(S_g) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$
$$X \mapsto (\ell_1(X), \dots, \ell_{3g-3}(X), \tau_1(X), \dots, \tau_{3g-3}(X))$$

is a homeomorphism.

Proof: Pretty intuitive/clear that the parameters l_i, τ_i are continuous

End 2/7 Exercise: Convince yourself of this!

Fri 2/9/18

Construct Inv: given $(l_1, \dots, l_{3g-3}, \tau_1, \dots, \tau_{3g-3})$, we build $X \in \mathcal{T}(S_g)$ with these parameters.

1) for each pants P of $S_g \setminus \bigcup \gamma_i$, say with bdry curves $\gamma_i, \gamma_j, \gamma_k$

Let X_P = hyp pair of pants with boundary lengths l_i, l_j, l_k

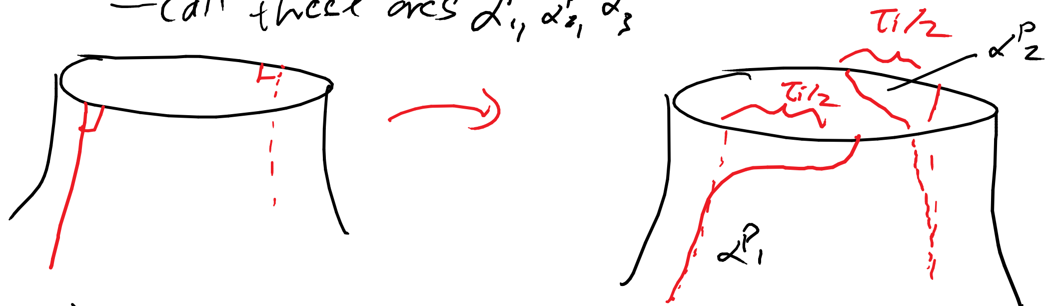
Know $\exists!$ (up to htpy) homeo $\psi_P : P \to X_P$ matching up boundaries correctly

2) For each X_i draw unique geod. arcs pairwise connecting boundaries

- for each boundary comp of X_P, say corresp to γ_i,

adjust these arcs (in nbhd of bdry) & replace with arcs that first travel (signed) dist $\tau_i/2$ along boundary before terminating

—call these arcs $\alpha_1^P, \alpha_2^P, \alpha_3^P$



Adjust homeos $\psi_P : P \to X_P$ by homeo to send 3 seam arcs in P to those 3 arcs α_i^P in X_P

Now: glue all hyp pants X_P together along geodesic boundary st
- endpts of α arcs match up
- maps $\psi_P : P \to X_P$ descend to quotient

$$\psi : S_g \to X = \bigsqcup X_P / \sim .$$

exists unique way to do this gluing & get hyp surf X

resulting map $\psi : S_g \to X$ unique up to homotopy

gives pt $(X, \psi) \in \mathcal{T}(S_g)$ whose FN parameters

are clearly $(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3})$ by construction

This proves FN is bijective.

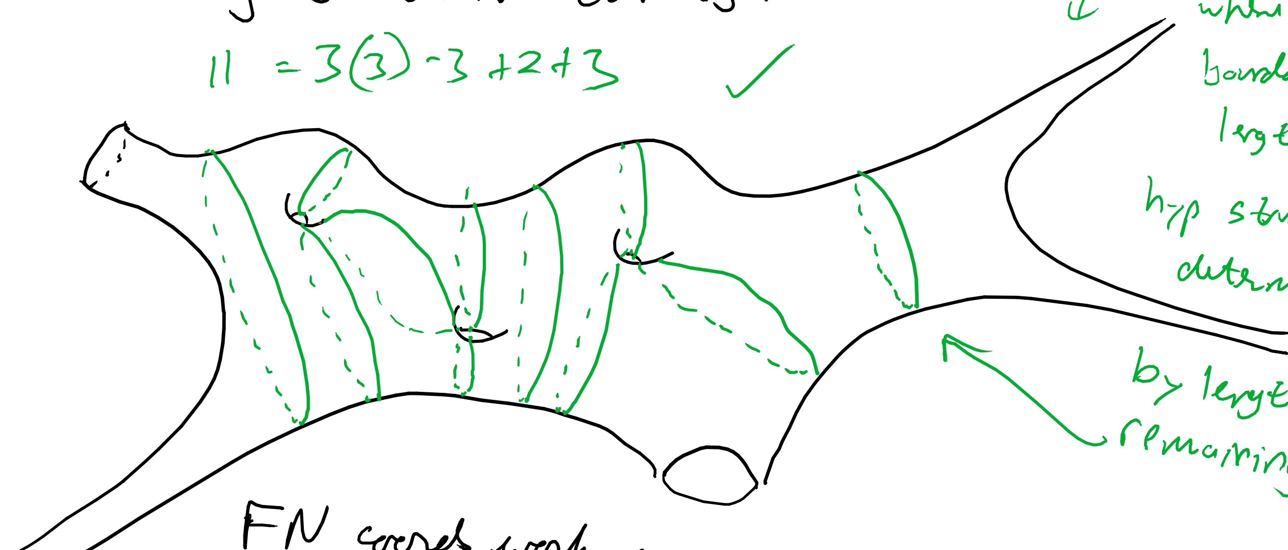
Again kind of clear that FN^{-1} is continuous ☒

Fenchel Nielsen Coords for non-closed surfaces:

A pants decomposition of $S_{g,n}^b$ has:

$3g - 3 + b + n$ curves:

$11 = 3(3) - 3 + 2 + 3$ ✓



a "degenerate pants" where 2 boundaries have length 0.

hyp structure determined by length of remaining boundary

FN coords work same way:

- each pants curve gets twist param + length param
- plus length parameter for each boundary curve.

$\rightsquigarrow$ $3g - 3 + 2b + n$ length params

$3g - 3 + b + n$ twist params

Aside: Exterior algebras, differential forms, wedge product:

A, B vect. spaces over $\mathbb{R}$.

set $F(A\times B)$ = free vect. space on set $A\times B$

R = subspace gen by all elts of form:

$$(a_1+a_2, b) - (a_1, b) - (a_2, b)$$
$$(a, b_1+b_2) - (a, b_1) - (a, b_2)$$
$$\lambda(a,b) - (\lambda a, b)$$
$$\lambda(a,b) - (a, \lambda b)$$

Tensor product is $A\times B = F(A\times B)/R$

coset of (a,b) denoted $a\otimes b$ (simple tensor)

e.g.: $A\otimes A$, , $\underbrace{A\otimes\cdots\otimes A}_{n} = \bigotimes_{i=1}^{n} A$.

Exterior product $A\wedge A = A\otimes A / \mathrm{span}\{a\otimes a \mid a\in A\}$

$$A\wedge\cdots\wedge A = \Lambda^n A = \otimes^n A / \mathrm{Span}\left\{a_1\otimes\cdots\otimes a_n \;\middle|\; \begin{array}{l} a_i = a_j \\ \text{some } i\neq j\end{array}\right\}$$

· Image of $a_1\otimes\cdots\otimes a_n$ denoted $a_1\wedge\cdots\wedge a_n$

note: in $A\wedge A$, $0 = (a+b)\wedge(a+b) = \cancel{a\wedge a} + b\wedge a + a\wedge b + \cancel{b\wedge b}$

$$\Rightarrow \quad b\wedge a = -a\wedge b$$

<u>Antisymmetrization map</u> $\mathrm{Ant}: \bigotimes^n A \to \bigotimes^n A$, def on simple tensors as

$$\mathrm{Ant}(a_1 \otimes \cdots \otimes a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

- $\ker(\mathrm{Ant}) = \{\mathrm{span}\ a_1 \otimes \cdots \otimes a_n \mid a_i = a_j \text{ some } i \neq j\}$ so

$$\Lambda^n A \cong \mathrm{Image}(\mathrm{Ant})! \quad \left(\text{may view } \Lambda^n A < \bigotimes^n A\right)$$

A, B vect spaces. A map

$f: A \oplus \cdots \oplus A \to B$ is:

- <u>multilinear</u> if linear in each factor (when other coords fixed)
- <u>alternating</u> if multilinear & $f(a_1, \dots, a_n) = 0$ whenever $\{a_1, \dots, a_n\}$ lin dependent ($\Longleftrightarrow$ whenever $a_i = a_j$ some $i \neq j$)

$A^* = \mathrm{Hom}(A, \mathbb{R})$ dual vector space.

<u>Exercise</u> If A, B fin dimensional:

1) $A^* \otimes B \cong \mathrm{Hom}(A, B)$ $\quad (\lambda \otimes b \mapsto (a \mapsto \lambda(a) b))$
2) $\bigotimes^k A^* \cong$ space of multilinear maps $A \oplus \cdots \oplus A \to \mathbb{R}$
3) $\Lambda^k A^* \cong$ space of alternating maps $A \oplus \cdots \oplus A \to \mathbb{R}$

★ under these identifications, viewing $\Lambda^k A^* \leq \bigotimes^k A^*$.

M smooth mnfld, T^*M cotangent bundle.

Recall: a 1-form is a section of $T^*M \to M$

<u>Def</u> A k-form on M is a (smooth) section of the bundle

$$(\Lambda^k T^*M) \to M$$

That's what a k-form is: smooth choice of alt. map on each tangent space.

$\Omega^k(M)$ = set of k-forms.

so: $\omega \in \Omega^k(M)$, then at each $p \in M$,

$$\omega_p \in \Lambda^k(T_p^* M) = \text{space alternating maps } \bigoplus^k T_pM \to \mathbb{R}$$

<u>Wedge Product</u> $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$

given $\omega \in \Omega^k(M)$, $\mu \in \Omega^l(M)$, need alt. map on $\bigoplus^{k+l} T_pM$:

view ω, μ as just multi. maps, in tensor product.

<u>Def</u> $\omega \wedge \mu = \dfrac{(k+l)!}{k!\,l!} \operatorname{Alt}(\omega \otimes \mu)$

so:

$$\boxed{\omega \wedge \mu(v_1, \dots, v_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \mu(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})}$$

E 2/9

Mon 2/12/18

<u>Back to Teich</u>: Consider $\mathcal{T}(S_{g,n})$.

Fixing pants curves + seams, get Fenchel-Nielsen parameters

$$l_i, \tau_i : \mathcal{T}(S_{g,n}) \to \mathbb{R}, \quad i = 1, \dots, 3g-3+n$$

- So far have only discussed topology on $\mathcal{T}(S_{g,n})$, but it can also be given smooth structure s.t. l_i, τ_i smooth

(i.e. give smooth structure from $\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$)

– This is well defined

3-24

Obviously the coordinates $X \mapsto (\ldots l_i(x), \ldots \tau_i(x), \ldots)$

depend on the chosen pants / seam curves!

get 1-forms dl_i, $d\tau_i$ (also depend on choices!)

form 2-form: $\sum_{i=1}^{3g-3+n} dl_i \wedge d\tau_i$

Wolperts Magic Formula

let l_i, τ_i be FN coords on $\mathcal{T}(S_{g,n})$ associated to

any pants decomp (+ seams). Then the 2-form

$$\omega = \frac{1}{2} \sum_{i=1}^{3g-3+n} dl_i \wedge d\tau_i$$

is canonical: does not depend on the choice of pants / seams!

Cor ω is invariant under the action $Mod(S) \curvearrowright \mathcal{T}(S)$:

$$\phi^*(\omega) = \omega \quad \text{for all } \phi \in Mod(S)$$

if $\mu \in \Omega^k(M)$ a k-form on M + $f: M \to M$ smooth,

get pull-back k-form $f^*(\mu)$:

$$(f^*\mu)_p(v_1, \ldots, v_k) = \mu_{f(p)}(D_pf(v_1), \ldots, D_pf(v_k))$$

proof: let $\gamma = \{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ + $\beta = \{\beta_1, \ldots, \beta_m\}$ be curve system of

curves for our coords l_i, τ_i. Then may need ϕ^{-1} here?

$\phi(\gamma) = \{\phi(\gamma_i)\}$, $\phi(\beta) = \{\phi(\beta_j)\}$ is also a curve sys of

curves, giving new coords $\tilde{l}_i$, $\tilde{\tau}_i$. Clearly

$$\phi^*(\omega) = \frac{1}{2} \sum_i d\tilde{l}_i \wedge d\tilde{\tau}_i \underset{\text{wolpert!}}{=} \frac{1}{2} \sum_i dl_i \wedge d\tau_i = \omega \quad \blacksquare$$

Wolpert proves this by showing $\frac{1}{2}\sum d\ell_i \wedge d\tau_i$ equals the <u>Weil Petersson</u> symplectic form:

<u>Complex Analysis</u>:

<u>Defn</u>: A <u>Riemann Surface</u> is a complex 1-mnfld.

That is, a topological space X equipped with an atlas of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ where

- $\{U_\alpha\}$ open cover of X
- $V_\alpha \subset \mathbb{C}$ open
- each φ_α a homeo
- transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \to V_\beta$ (where defined) are biholomorphic (holomorphic with holomorphic inverse)

  i.e, Complex diff: satisfy Cauchy-Riemann eqns.

If X is a Riem Surf, then univ cover $\tilde{X}$ is a simply connected Riemann Surface on which deck group $\pi_1(X)$ acts by conformal (i.e biholomorphic) automorphisms.

<u>Uniformization Theorem</u>:

Every simply connected Riem. surf is conf. equiv. (i.e biholom.) to one of the following:

- Riem. sphere $\hat{\mathbb{C}}$
- complex plane $\mathbb{C}$
- unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$

Note: · The conf. automorphism group of $\hat{\mathbb{C}}$

$= $ group of Möbius transfs $= PSL(2,\mathbb{C})$

· $\mathbb{H}^2 = \{ z \mid \operatorname{Im}(z) > 0\}$ + $D = \{ u \mid |u| < 1\}$ are conf equiv via

$$f(z) = \frac{z-i}{-iz+1} \qquad \infty \to i \quad i \to 0 \qquad 0 \to -i \quad 1 \to 1$$

$$\begin{tikzcd} \mathbb{H}^2 \arrow[r, bend left, "f"] & D \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\frac{u+i}{iu+1} = g(u) \qquad 0 \to i \quad i \to \infty \qquad 1 \to 1 \quad -i \to 0$$

· Conf autom group of $\mathbb{H}^2 = PSL(2,\mathbb{C}) \cap$ Stabilizer of $\mathbb{H}^2 = PSL(2,\mathbb{R})$
(similarly for D).

· Conf autom group of $\mathbb{C} = \{ z \mapsto az+b \mid b \in \mathbb{C},\ a \in \mathbb{C}^* \}$

$= \mathbb{C}^* \rtimes \mathbb{C}$ (rotate, scale, + translate)

Notice: with our defns, every hyp surf is naturally a Riem. surface

pf: Let X be hyp surf, so $X = \mathbb{H}^2/\Gamma$, some $\Gamma < PSL(2,\mathbb{R})$ acting freely + prop disc. $\mathbb{H}^2 \to \mathbb{H}^2/\Gamma = X$ is covering map, so for each $p \in X$, may choose nbhd $p \in U \subset X$ that is evenly covered + hence $\varphi_p : U_p \to V_p$, where $V \subset \mathbb{H}^2 \subset \mathbb{C}$ is component of preimage of U.
Then atlas $\{ \varphi_p : U_p \to V_p \}$ defines Riem. Surf since transition maps given by elts of Γ (which are biholomorphic!) ☒

Converse to this follows from Uniformization:

Conseq of Uniformization:

Every Riem surf admits a Riem. metric of const curvature either 1, 0, or -1 dep. on whether Euler characteristic is +, 0, or −.

Inducing same conf structure!

proof: Let X be R.S. By unif, may identify $\tilde{X}$ with $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}^2$ s.t. deck group $\pi_1(X) = \Gamma$ acts freely & prop disc. by conf. automorphisms.

- If $\tilde{X} = \hat{\mathbb{C}}$, must have $X = \tilde{X} = \hat{\mathbb{C}}$
  (no proper manifold quotients of sphere: no subgrp of $PSL(2,\mathbb{C})$ acts freely — every elt has 2 fixed pts!!)
  so give X std metric on $\hat{\mathbb{C}} = 2$-sphere (curv 1)

- If $\tilde{X} = \mathbb{C}$, $\Gamma \leq \mathbb{C}^* \ltimes \mathbb{C}$ act free & prop disc $\Rightarrow \Gamma \cap \mathbb{C}^* = \{1\}$
  here Γ acts by translations
  $\rightarrow$ preserve std metric on $\mathbb{C} = \mathbb{R}^2$ (curv 0)

- If $\tilde{X} = \mathbb{H}^2$, evidently $X = \mathbb{H}^2/\Gamma$ for some $\Gamma < PSL(2,\mathbb{R})$ acting freely & prop disc. & preserving hyp metric on $\mathbb{H}^2$
  so X is a hyp surf & gets metric of (curv −1)

Gauss Bonnet Thm $\Rightarrow$ sign of any const curvature metric must agree w/ sign of Euler char. $\boxed{\text{☒}}$

$$\int_X \text{curv} = 2\pi \chi(X)$$

— End 2/12/18

Wed 2/14/18

Upshot: alternate defn of Teichmuller space:

$$\mathcal{T}(S_{g,n}) = \left\{ (X,f) \;\middle|\; \begin{array}{l} X \text{ a Riem. surf \&} \\ f: S_{g,n} \to X \text{ homeo} \end{array} \right\} \Big/ \sim$$

where $(X,f) \sim (Y,g)$ if $g \circ f^{-1}$ isotopic to biholomorphism.

Quadratic Differentials.

The <u>holomorphic cotangent bundle</u> of a R.S. X is (complex) line bundle

$$\begin{array}{c} K \\ \downarrow \\ X, \end{array}$$

fiber over $p \in X$ is space of complex linear maps $T_pX \to \mathbb{C}$

– has a natural complex structure

A <u>holomorphic 1-form</u> on X is a holomorphic section of K.

<u>Def</u> A (holomorphic) <u>quadratic differential</u> on X is a holom. section of $\mathrm{Sym}^2(K)$ = symmetric square of K

$\left(V \text{ vect space. } \mathrm{Sym}^2(V) = V \otimes V / \mathrm{span}\{ v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V \} \right)$

Space of quadratic differentials on X forms a $\mathbb{C}$-vector space

<u>def:</u> $QD(X)$

<u>Fact</u> (consequence of Riemann Roch thm)

$$\dim_{\mathbb{C}} (QD(X)) = 3g - 3 \quad \text{if } X \cong S_g$$

Quad Diffs in local coordinates

Let $\{ z_\alpha : U_\alpha \to \mathbb{C} \}$ be atlas of charts on X

a holom quad diff on X is given by collection of expressions $\{ \varphi_\alpha(z_\alpha)\, dz_\alpha^2 \}$, where:

1) each $\varphi_\alpha : z_\alpha(U_\alpha) \to \mathbb{C}$ is holom with finitely many zeros

2) for any 2 charts z_{α}, z_{β} have

$$\varphi_\beta(z_\beta) \left(\frac{dz_\beta}{dz_\alpha} \right)^2 = \varphi_\alpha(z_\alpha)$$

$\frac{d\psi}{dz}$ if $\psi : z_\alpha(U_\alpha) \to z_\beta(U_\beta)$ is transition funct.

$\boxed{3-2g}$

Interpret this as: collection $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ invariant under change of local coordinates.

A quad. diff on X gives a holomorphic map

holom tangent bundle of $X \longrightarrow \mathbb{C}$

So, in local coords quad diff q given by $\phi(z)dz^2$

+ tangent vect v by $\alpha \in T_{z_0}\mathbb{C} \simeq \mathbb{C}$

then $\boxed{q(v) = \phi(z_0)\alpha^2}$

Many many things one could say about + do with quadratic differentials. For us point is:

Given top surf S, get vector bundle

$$\begin{array}{c} QD(S) \\ \downarrow \\ \mathcal{T}(S) \end{array}$$

where fiber over $X \in \mathcal{T}(S)$ is vect space $QD(X)$ of quad. diffls on X.

<u>non-obvious fact</u>: bundle $QD(S)$ is naturally isomorphic to cotangent bundle $T^*(\mathcal{T}(S))$ of Teich.

So: can define a Riemannian cometric on Teich by giving a pairing on $QD(X)$:

The Weil-Petersson pairing:

Let X be a Riem. Surf., say with local coord charts $\{z_\alpha\}$

know X can be equipped with unique hyp. metric ρ.

- in coordinates $z_\alpha = x_\alpha + i y_\alpha$, has form $\rho(z_\alpha)|dz_\alpha|$

i.e. $\rho(z_\alpha)\cdot$(Euclidean inner prod at z_α).

Say $\varphi, \psi \in QD(X)$, in local coords:

$$\varphi_\alpha(z_\alpha)\, dz_\alpha^2, \quad \psi_\alpha(z_\alpha)\, dz_\alpha^2.$$

Define:

$$\langle \varphi, \psi \rangle_{WP} = \int_X \frac{\varphi \overline{\psi}}{\rho^2} \overset{\text{in coords}}{=} \int_X \frac{\varphi_\alpha(z_\alpha)\overline{\psi(z_\alpha)}}{\rho(z_\alpha)^2}\, dArea(z_\alpha)$$

This defines a Hermitian inner product on $QD(X)$

(lin in first coord, $\overline{\langle \psi, \varphi\rangle} = \langle \varphi, \psi \rangle$, pos def.)

write as

$$\langle \cdot, \cdot \rangle_{WP} = \mathrm{Re}\langle \cdot, \cdot \rangle_{WP} + i\, \mathrm{Im}\langle \cdot, \cdot \rangle_{WP}$$

Then $\mathrm{Re}\langle \cdot, \cdot \rangle_{WP}$ is an inner product on $QD(X)$

$\rightsquigarrow$ defines a Riem. metric on $\mathcal{T}(S)$!!

The "Weil-Petersson" metric.

also have the imaginary part:

$\omega = \mathrm{Im}\langle \cdot, \cdot \rangle_{WP}$. This is a bilinear, antisymmetric pairing, i.e.

ω is an alternating map $QD(X)^2 \to \mathbb{R}$.

Hence: ω defines a 2-form on $\mathcal{T}(S)$

The "Weil-Petersson form".

Fact: WP Hermitian product is "Kähler", meaning that exterior derivative $d\omega = 0$

$d: \Omega^k \to \Omega^{k+1}$ def by:

1) for f smooth funct, df = differential, $d(df) = 0$

2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for α a p-form

Thm (Wolpert) This 2-form ω is equal to

$$\frac{1}{2} \sum_i dl_i \wedge d\tau_i$$

for any FN-parameters l_i, τ_i

Brief sketch:

fix α any scc on S. Def FN twist vector field t_α on $\mathcal{T}(S)$ as follows:

at $X \in \mathcal{T}(S)$, get path $\gamma: \mathbb{R} \to \mathcal{T}(S)$

$\gamma(\theta)$: cut X on geod α, twist hyp dist θ to right (calibrated by displacement of reference pts) + reglue.

set $\boxed{t_\alpha(X) = \gamma'(0)}$

Key Ingredient: Twist-Length Duality

for any scc α on S,

$$\boxed{2\omega(\cdot, t_\alpha) = dl_\alpha}$$

(both are forms; eat tangent vectors)

from this, basic symplectic geom stuff $\Rightarrow$ ω invariant under the twist flows.

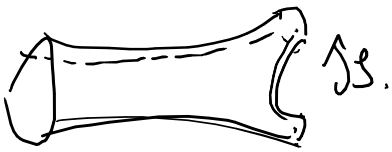
Now fix coords l_i, τ_i + write $\omega \in \text{span}\{dl_i \wedge d\tau_j, d\tau_i \wedge d\tau_j, dl_i \wedge dl_j\}$

Check coefficients:

coeff of $dl_i \wedge d\tau_j$ is $\omega\left(\frac{\partial}{\partial l_i}, \frac{\partial}{\partial \tau_j}\right) \overset{\text{duality}}{=} \frac{1}{2}\, dl_j\left(\frac{\partial}{\partial l_i}\right) = \frac{1}{2}\delta_{ij}$

for other coefficients:

each pair of pants has involution ρ : exchange hexagons



Adjust twist params τ_i to get to a pt where ρ assembles to global reflection of surf.

↳ ω twist invariant, so this does not change the coeff of the 2-form ω.

wrt ρ: ω is odd, dl_i invariant

$\Rightarrow$ coeff $dl_i \wedge dl_j$ must $= 0$!

ω odd & each $d\tau_j$ mod dl_j is odd

$\Rightarrow$ coeff $d\tau_i \wedge d\tau_j = 0$! $\boxed{\text{Fin}}$

E 2/14

Fri 2/16/18

Upshot: The 2-form $\omega = \frac{1}{2}\sum_i dl_i \wedge d\tau_i$ is canonical

- indep of FN-coord system
- invariant under Mod(S)-action on Teich
  - ↳ descends to a 2-form on quotient = moduli space
- ω is symplectic, so $\omega^{3g-3+n} = \omega\wedge\cdots\wedge\omega$ is nowhere vanishing top dim form = a volume form on $\mathcal{T}(S)$.
  (in coords: $2^{-(3g-3+n)}\, dl_1 \wedge d\tau_1 \wedge \cdots \wedge dl_{3g-3+n} \wedge d\tau_{3g-3+n}$)

we will see that, with this volume form, quotient Moduli space has finite volume!

$\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$ preserving the volume form

quotient is <u>moduli space</u> $\mathcal{M}(S)$

<u>Bers</u> $\exists$ const $b = b(S)$ (dep only on top of finite type surf S)

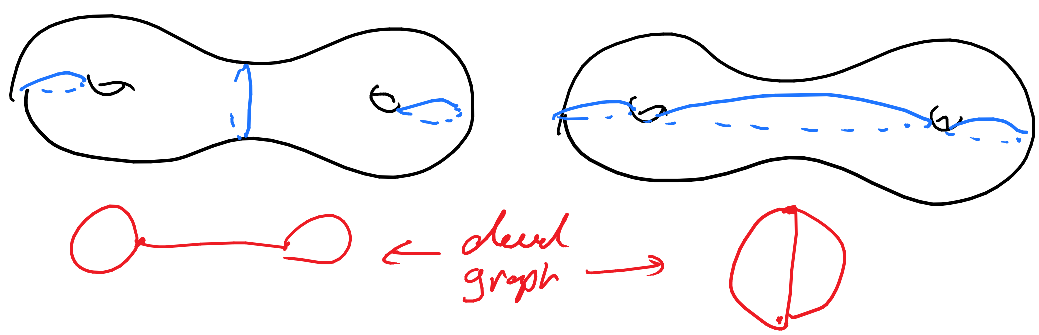
s.t. any $X \in \mathcal{T}(S)$ has a pants decomposition

$\gamma = \{\gamma_i\}$ s.t. $\ell_X(\gamma_i) \leq b \quad \forall i$

<u>Cor</u>: The WP-volume of moduli space is finite.

<u>proof</u>: exist only finitely many pants decompositions up to orbit of mapping class group

Determined by combinatorial data of how pants are glued:



If dual graphs agree, $\exists$ homeo sending one to the other — use classification of surfaces!

Choose pants decompositions $\gamma_1, \ldots, \gamma_k$, one for each orbit

$\gamma_i = \{\gamma_{i1}, \ldots, \gamma_{i,3g-3+n}\}$

For each, get FN coords $\mathcal{T}(S) \xrightarrow{FN_i} (\ldots \ell_{ij}, \ldots, \tau_{ij})$

Set $D_i = \{\ell_{ij} \leq b,\ 0 \leq \tau_{ij} \leq \ell_{ij}\} \subset \mathcal{T}(S)$ in FN_i coords.

Bers fact $\Rightarrow$ each orbit $\mathrm{Mod}(S) \cdot X$ in $\mathcal{T}(S)$ intersects

$D_1 \cup \cdots \cup D_k$ finite <u>positive</u> # of times.

formula for ω $\Rightarrow$ each D_i has finite volume

$\Rightarrow \mathcal{M}(S)$ has finite volume! $\blacksquare$