II. The Hyperbolic Plane

<u>Def</u> The hyperbolic plane is the smooth 2-mnfld

$$\mathbb{H}^2 = \underbrace{\{(x,y) \in \mathbb{R}^2 \mid y > 0\}}_{\text{upper half plane}}$$

equipped with Riem. metric.

open set in mnfld $\mathbb{R}^2$

Single chart

$\{ id : \mathbb{H}^2 \to \mathbb{H}^2 \subset \mathbb{R}^2 \}$

$\langle \cdot , \cdot \rangle_{(x,y)} = \dfrac{g_{(x,y)}}{y^2}$ on $T_{(x,y)} \mathbb{H}^2$, when g is std. metric on $\mathbb{R}^2$

so wrt basis $\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_2}\right)_p \right\}$ of $T_p \mathbb{H}^2$ (from chart)

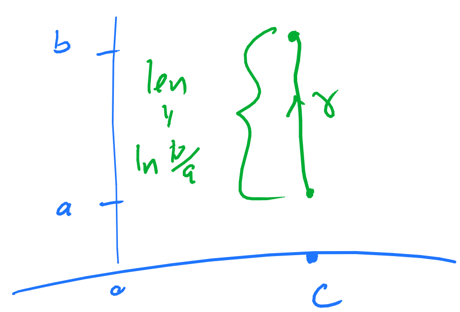
where $p = (x,y)$

$$\left\langle \sum_{i=1}^{2} a_i \left(\frac{\partial}{\partial x^i}\right)_p, \sum_{i=1}^{2} b_i \left(\frac{\partial}{\partial x^i}\right)_p \right\rangle_p = \frac{a_1 b_1 + a_2 b_2}{y^2}$$

Example paths / lengths

1) for $0 < a < b$, $c \in \mathbb{R}$, consider path

$\gamma : [a,b] \to \mathbb{H}^2$, $\quad \gamma(t) = (c, t)$.



wrt basis vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$,

$$\gamma'(t) = 0 \frac{\partial}{\partial x_1} + 1 \frac{\partial}{\partial x_2}.$$

$$\| \gamma'(t) \| = \sqrt{\frac{0 \cdot 0 + 1 \cdot 1}{t^2}} = \frac{1}{t}$$

so $l_{\mathbb{H}^2}(\gamma) = \int_a^b \| \gamma'(t) \| \, dt = \int_a^b \frac{1}{t} = \boxed{\ln(b/a)}$

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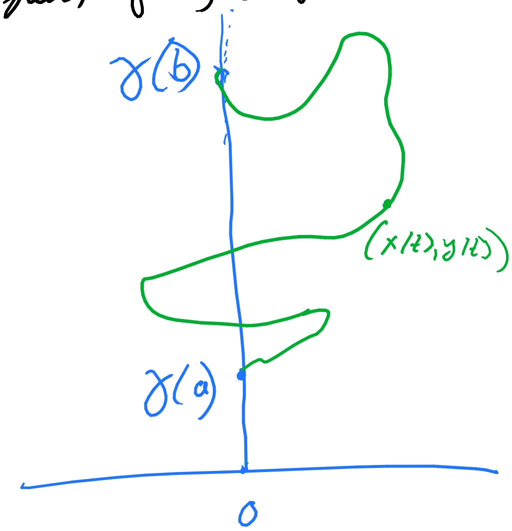
<u>claim</u> for any $c \in \mathbb{R}$, the path

$$\gamma : (0,\infty) \to \mathbb{R}$$
$$t \mapsto (c,t) \text{ is a geodesic.} \quad (\text{infinite length!})$$

<u>pf</u>: Suffices to show: for any $0 < a < b < \infty$,

$\gamma|_{[a,b]}$ is length minimizing path from $\gamma(a)$ to $\gamma(b)$:



Let $u : [d,e] \to \mathbb{H}^2$ be such path.

write $u(t) = (x(t), y(t))$

$$u'(t) = x'(t)\frac{\partial}{\partial x_1} + y'(t)\frac{\partial}{\partial x_2}$$

$$\|u'(t)\| = \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{(y(t))^2}}$$

$$\geq \sqrt{\left(\frac{y'(t)}{y(t)}\right)^2} \geq \frac{y'(t)}{y(t)}. \quad \text{so}$$

$$l_{\mathbb{H}^2}(u) = \int_d^e \|u'(t)\|\,dt \geq \int_d^e \frac{y'(t)}{y(t)}\,dt = \ln|y(t)|\Big]_d^e$$

$$= \ln\left|\frac{y(e)}{y(d)}\right| = \ln(b/a) = l_{\mathbb{H}^2}(\gamma|_{[a,b]}) \quad \checkmark$$

Thus γ is a geodesic, as claimed $\boxtimes$

Notice: get equality $\iff$

- $x'(t) \equiv 0$
- $y'(t) > 0$ (y monotone)

Thus γ is <u>unique</u> shortest path.

2) for $r>0$, $c \in \mathbb{R}$, $0 < \theta_1 < \theta_2 < \pi$

path $\alpha : [\theta_1, \theta_2] \to \mathbb{H}^2$

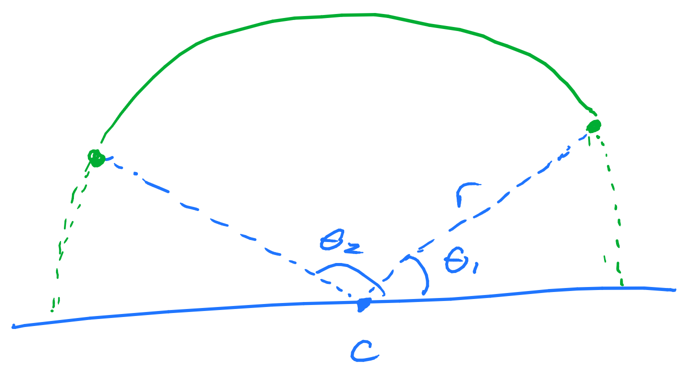
$$\alpha(t) = \left(c + r\cos(t),\ r\sin(t) \right)$$

$$\alpha'(t) = -r\sin(t)\frac{d}{dx_1} + r\cos(t)\frac{d}{dx_2}$$

so

$$\|\alpha'(t)\|^2 = \frac{r^2\sin^2(t) + r^2\cos^2(t)}{r^2\sin^2(t)}$$

$$\boxed{\|\alpha'(t)\| = \frac{1}{\sin(t)}}$$



$$\text{so: } \ell_{\mathbb{H}^2}(\alpha) = \int_{\theta_1}^{\theta_2} \frac{1}{\sin(t)}\,dt = \int_{\theta_1}^{\theta_1} \frac{\csc t\,(\csc t + \cot t)}{\csc t + \cot t}\,dt$$

$$= \int_{\theta_1}^{\theta_2} \frac{-\frac{d}{dt}(\csc t + \cot t)}{\csc t + \cot t}\,dt = -\ln\left|\csc t + \cot t\right|\Big]_{\theta_1}^{\theta_2}$$

$$= \ln\left| \frac{\csc\theta_1 + \cot\theta_1}{\csc\theta_2 + \cot\theta_2} \right|$$

- doesn't depend on r!
- goes to ∞ as $\theta_1 \to 0$ or $\theta_2 \to \pi$.

Exercise: Show that α is a geodesic via a semi computation.

(write $u(t) = (r(t)\cos\theta(t),\ r(t)\sin(\theta(t))$.

calculate $\|u'(t)\|$ & estimate from below).

$$\gamma(t) = (c + r(t)\cos\theta(t),\ r(t)\sin\theta(t)) \qquad t \in [a,b]$$

$$\gamma'(t) = (r'\cos\theta - r\sin\theta\,\theta')\frac{\partial}{\partial x_1} + (r'\sin\theta + r\cos\theta\,\theta')\frac{\partial}{\partial x_2}$$

$$\|\gamma'(t)\|^2 = \frac{(r')^2\cos^2\theta \cancel{-2rr'\theta'\cos\theta\sin\theta} + r^2(\theta')^2\sin^2\theta + (r')^2\sin^2\theta \cancel{+2rr'\theta'\cos\theta\sin\theta} + r^2(\theta')^2\cos^2\theta}{r^2\sin^2\theta}$$

$$\|\gamma'(t)\|^2 = \frac{(r')^2 + r^2(\theta')^2}{r^2\sin^2\theta} \geq \frac{(\theta')^2}{\sin^2\theta}.$$

Think about when you got equality.

so $$\ell_{\mathbb{H}^2}(\gamma) = \int_a^b \|\gamma'(t)\|\,dt \geq \int_a^b \frac{\theta'(t)}{\sin\theta(t)}\,dt$$

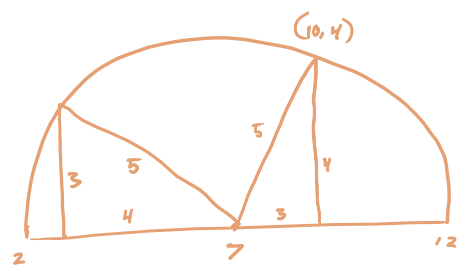
$$= \int_{\theta(a)=\theta_1}^{\theta(b)=\theta_2} \frac{1}{\sin\theta}\,d\theta = \ell_{\mathbb{H}^2}(\delta) \quad \checkmark$$

Observe

1) We have now found a geodesic between any 2 pts of $\mathbb{H}^2$

 – any 2 pts with distinct x-coordinates lie on a semi-circle centered on x-axis.

 Exercise Find geodesic through $(3,3)$ + $(10,4)$



2) We have found biinfinite (= infinite length in both directions) geodesic through any tangent vector.

Fact that a geodesic is locally determined by a tangent vector
$\Rightarrow$

- These are all of the geodesics in $\mathbb{H}^2$
- There is a unique geod connecting any 2 pts

} up to reparametrization.

— See that every geod. may be extended indefinitely.

Möbius Transformations

Identify $\mathbb{R}^2 = \mathbb{C}$, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\}$
one pt compactification.

Group $SL(2,\mathbb{C}) \curvearrowright \hat{\mathbb{C}}$ by Möbius transformations:

$$\underset{\underset{SL(2,\mathbb{C})}{\cap}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} z = \frac{az+b}{cz+d} \quad \begin{pmatrix} = a/c \text{ if } z=\infty \\ = \infty \text{ if } cz+d=0 \end{pmatrix}$$

— check: a group action!

Lem If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ & $z \in \mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\} \subset \hat{\mathbb{C}}$,
then $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}(z) / \|cz+d\|^2$.

Proof:

$$\underbrace{\frac{az+b}{cz+d}}_{\substack{\text{nonzero since} \\ z \neq -d/c}} = \frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}} = \frac{\overbrace{ac\|z\|^2 + bd}^{\in \mathbb{R}} + adz + bc\bar{z}}{\underbrace{\|cz+d\|^2}_{>0}}$$

So:

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im}(adz+bc\bar{z})}{\|cz+d\|^2} = \frac{(adz+bc\bar{z}) - \overline{(adz+bc\bar{z})}}{2\|cz+d\|^2}$$

$$= \frac{(ad-bc)z + (bc-ad)\bar{z}}{2\|cz+d\|^2} = \frac{z-\bar{z}}{2\|cz+d\|^2} = \frac{\operatorname{Im}(z)}{\|cz+d\|^2} > 0 \quad \blacksquare$$

End 1/22/18

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Wed 1/24/18

Cor Each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$,

$$z \mapsto A\cdot z = \frac{az+b}{cz+d} \quad \text{gives bijection } \mathbb{H}^2 \to \mathbb{H}^2.$$

proof: Mapping $A: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ bijective since it is a group action.

So $A|_{\mathbb{H}^2}$ injective.

for $z \in \mathbb{H}^2$, $A^{-1}z \in \mathbb{H}^2$ (by lem), so

$z = A(A^{-1}z)$ in Range of $A|_{\mathbb{H}^2}$. So onto

(note: $\infty \mapsto a/c \in \mathbb{R}\cup\{\infty\}$, so $\notin \mathbb{H}^2$, $\mathbb{R}\cup\{\infty\} \ni -\frac{d}{c} \mapsto \infty$) ∎

Prop for $A \in SL(2,\mathbb{R})$, map $A: \mathbb{H}^2 \to \mathbb{H}^2$,
$z \mapsto A\cdot z$, is an <u>isometry</u> of Riem mnfld $\mathbb{H}^2$.

Proof:

<u>Linear Algebra Fact</u>: If $T: V \to W$ is an isomorphism of f-dim'l real inner product spaces s.t. $\|T(v)\| = \|v\|$ $\forall v \in V$, then $\langle T(v), T(w)\rangle = \langle v, w\rangle$ $\forall v,w \in V$ (i.e. T an isometry)

$$\|v+w\|^2 = \|v\|^2 + 2\langle v,w\rangle + \|w\|^2$$
$$\|v-w\|^2 = \|v\|^2 - 2\langle v,w\rangle + \|w\|^2, \quad \text{so}$$
$$\langle v,w\rangle = \frac{\|v+w\|^2 - \|v-w\|^2}{4}$$

Let $z \in \mathbb{H}^2$, $v \in T_z\mathbb{H}^2$ arbitrary. Suffices to show:

A diff at z, +

$\|D_z A(v)\| = \|v\|$

$A: \mathbb{H}^2 \to \mathbb{H}^2$ is complex differentiable: $A(z) = \dfrac{az+b}{cz+d}$.
$\quad\ \ \cap \qquad \cap$
$\quad\ \ \mathbb{C} \qquad \mathbb{C}$

$$A'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{acz - acz + ad - bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

so it is holomorphic and smooth.

Let $v \in T_z \mathbb{H}^2$, choose $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{H}^2$, $\left.\begin{array}{l}\gamma(0) = z \\ \gamma'(0) = v\end{array}\right]$ Identify $T_z\mathbb{H}^2 = \mathbb{R}^2 = \mathbb{C}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \in$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \gamma'(0)$

$$D_z A(v) = \frac{d}{dt}\left(A(\gamma(t))\right)\Big|_{t=0} = A'(\gamma(0)) \cdot \gamma'(0) = \frac{\gamma'(0)}{(cz+d)^2}$$

$$\text{So } \|D_z A(v)\| = \frac{\left|\gamma'(0)/(cz+d)^2\right|}{\operatorname{Im}(A \cdot z)} = \frac{|\gamma'(0)| / \cancel{\|cz+d\|^2}}{\operatorname{Im}(z) / \cancel{\|cz+d\|^2}} = \frac{|\gamma'(0)|}{\operatorname{Im}(z)} = \|v\|.$$

Hence A is an isometry, $\boxtimes$

Rmks
1) Identifying $T_z \mathbb{H}^2 \cong \mathbb{C}$, $T_{Az}\mathbb{H}^2 = \mathbb{C}$, see that $D_z A$ given by multiplication by complex number $\dfrac{1}{(cz+d)^2}$
$\hookrightarrow$ A preserves orientation of $\mathbb{H}^2$

2) get group homomorphism $SL(2,\mathbb{R}) \longrightarrow \operatorname{Isom}^+(\mathbb{H}^2) \leq \operatorname{Isom}(\mathbb{H}^2)$ orientation preserving
$\qquad\qquad\qquad\qquad\qquad\qquad\quad A \longmapsto (z \mapsto A \cdot z)$

kernel $= \{I, -I\}$ Indeed

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot z = \frac{-z+0}{0-1} = z.$$

conversely; $\dfrac{az+b}{cz+d} = z \quad \forall z \in \mathbb{H}^2$

$$az + b = cz^2 + dz$$

$$cz^2 + (d-a)z + b = 0 \quad \forall z$$

$\Rightarrow c = b = 0$, $d = a$, + $1 = ad - bc = ad$,

so $a = d \in \{1, -1\}$

$\overline{z\text{-}7}|$

Homomorphism descends to:

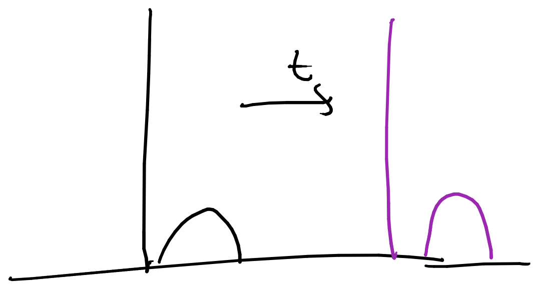
$$\mathrm{SL}(2,\mathbb{R}) / \{\pm I\} = \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Isom}^+(\mathbb{H}^2).$$

(i.e., action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ by isometries)

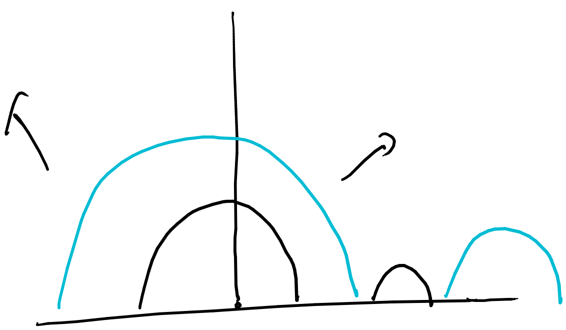
Example Isometries:

1) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $t \in \mathbb{R}$

translate horiz by t



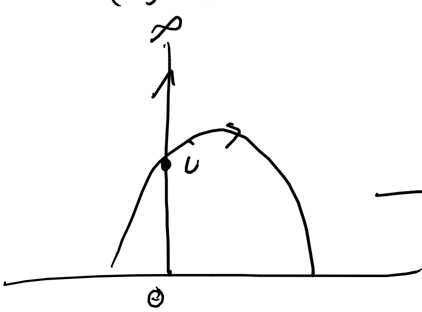
2) $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ $\lambda > 0$: Scale by λ:

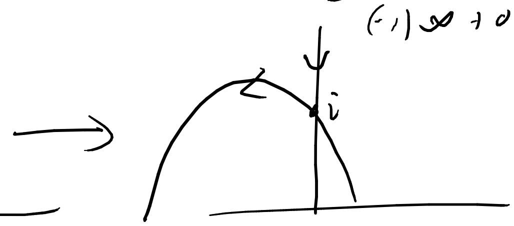


3) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: fix i & rotate by π:

$$i \mapsto \frac{0\,i + 1}{(-1)i + 0} = \frac{1}{-i} = i, \qquad 0 \mapsto \frac{0 \cdot 0 + 1}{(-1) \cdot 0 + 0} = \infty$$

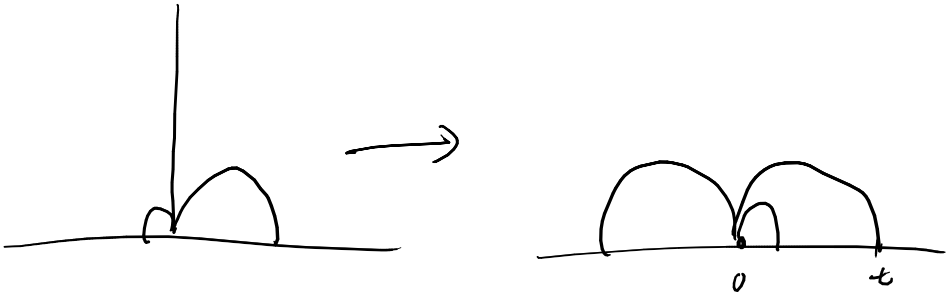
$$\infty \mapsto \frac{0 \cdot \infty + 1}{(-1)\infty + 0} = 0$$





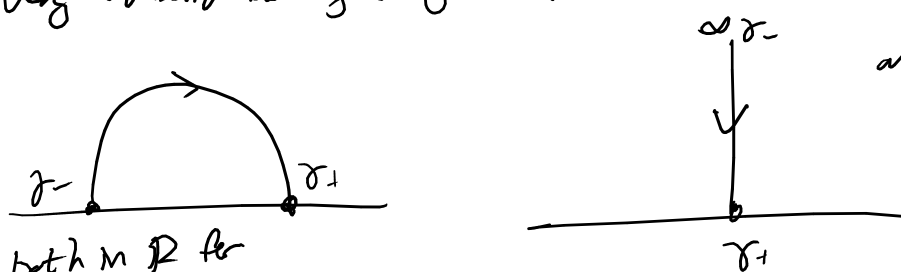
4) $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ $\quad 0 \mapsto \frac{0+0}{0+1} = 0, \quad \infty \mapsto \frac{1}{t} \quad (t \neq 0)$

everything else on $\mathbb{R}$ shifts to left $(t>0)$



More on Geodesics:

Every biinfinite geod γ in $\mathbb{H}^2$ has 2 distinct endpoints in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$



both in $\mathbb{R}$ for half-circle geod

one in $\mathbb{R}$, one $= \infty$ for vertical geod.

orientation on geod $\rightsquigarrow$ distinguish endpts γ^- γ^+

Observe: bijection

$$\mathcal{G} = \left\{ \begin{array}{c} \text{set of oriented} \\ \text{biinfinite} \\ \text{geodesics} \end{array} \right\} \longleftrightarrow \begin{array}{c} \text{ordered pairs of} \\ \text{distinct pts in } \hat{\mathbb{R}} \end{array}$$

$$\gamma \longmapsto (\gamma_-, \gamma_+)$$

unique geod w/ those endpts $\longleftarrow$ (a, b)

Note: $\mathrm{Isom}(\mathbb{H}^2) \curvearrowright \mathcal{G}$

$\uparrow$

$\mathrm{PSL}(2, \mathbb{R})$

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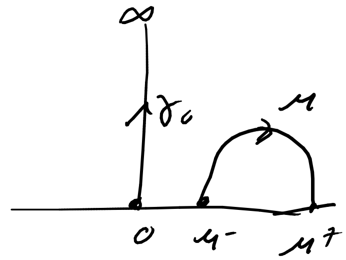
Prop $PSL(2,\mathbb{R})$ acts transitively on $\mathcal{G}$:

$\forall \gamma, \mu \in \mathcal{G}$, $\exists A \in PSL(2,\mathbb{R})$ st $A\gamma = \mu$

proof: Let γ_0 be geod $(0,\infty) \to \mathbb{H}^2$
$t \mapsto (0,t)$



Fix $\mu \in \mathcal{G}$.

suffices to show: $\exists A$ st $A\mu = \gamma_0$

Case 1 $\mu^- = \infty$.

Then $\begin{pmatrix} 1 & -\mu^+ \\ 0 & 1 \end{pmatrix} \cdot \mu$ has endpts $(\infty, 0)$

& $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\mu^+ \\ 0 & 1 \end{pmatrix} \mu$ has endpts $(0,\infty)$, so $= \gamma_0$

Case 2 $\mu^- \neq \infty$:

set $\mu_1 = \begin{pmatrix} 1 & -\mu^- \\ 0 & 1 \end{pmatrix} \mu$, endpts $(0, c)$

if $c = \infty$ done $(\mu_1 = \gamma_0)$

else, $\begin{pmatrix} 1 & 0 \\ -1/c & 1 \end{pmatrix} \mu_1$ has endpts $\left(0, \dfrac{c+0}{-1/c \cdot c + 1} = \infty\right)$

so $= \gamma_0$ ∎

via derivative, also get action $PSL(2,\mathbb{R}) \curvearrowright T\mathbb{H}^2$

$$A \cdot v = DA(v) = D_p A(v) \text{ for } v \in T_p \mathbb{H}^2$$

<u>Thm</u> $PSL(2,\mathbb{R})$ acts simply transitively on $T^1\mathbb{H}^2 = \{v \in T\mathbb{H}^2 \mid \|v\| = 1\}$

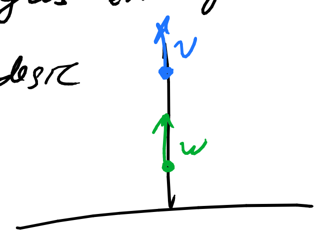
That is: $\forall v, w \in T^1\mathbb{H}^2$, $\exists!\, A \in PSL(2,\mathbb{R})$ s.t. $A \cdot v = w$.

proof:

<u>Existence</u>: Given v, w, let $\gamma, \mu \in \mathcal{G}$ be oriented geods through v, w.

find $A, B \in PSL(2,\mathbb{R})$ s.t. $A\gamma = B\mu = \gamma_0$ vert geodesic

Then $A \cdot v$ & Bw both unit vectors along γ_0



End 1/24 may choose $\lambda > 0$ s.t. $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} Av = Bw$ ✓

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<u>Uniqueness</u>: Suppose $A, B \in PSL(2,\mathbb{R})$ satisfy $Av = w = Bv$.

Then $B^{-1}Av = v$ set $p = \pi(v)$

In 2-dim vect. space $T_p\mathbb{H}^2$, $\exists!$ vect v' s.t.

$\{v, v'\}$ is oriented orthonormal basis

$B^{-1}A$ gives orientation preserving isom of $T_p\mathbb{H}^2$ respecting inner prod.

hence $B^{-1}A\{v, v'\}$ also oriented ONB

$B^{-1}Av = v$ + Uniqueness v' $\Rightarrow$ $B^{-1}Av' = v'$

Hence $B^{-1}A$ is identity on $T_p\mathbb{H}^2$

Now let $q \in \mathbb{H}^2$ arb, let $\gamma : [0, L] \to \mathbb{H}^2$ be unit speed geod w/ $\gamma(0) = p$, $\gamma(L) = q$.

Then $\gamma'(0) \in T_p\mathbb{H}^2 \Rightarrow B^{-1}A\gamma'(0) = \gamma'(0)$

$\Rightarrow B^{-1}A \cdot \gamma$ is a unit speed geod w/ initial vector $\gamma'(0)$ $\Rightarrow B^{-1}A\gamma = \gamma$ $\Rightarrow B^{-1}Aq = q$. $B^{-1}A = Id$ ∎

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Cor $PSL(2,\mathbb{R}) \cong Isom^+(\mathbb{H}^2)$

Proof: We have seen $PSL(2,\mathbb{R}) \to Isom^+(\mathbb{H}^2)$ injective.

For surjectivity, fix $v_0 \in T^1(\mathbb{H}^2)$ & let $\phi \in Isom^+(\mathbb{H}^2)$ arbitrary.

find $A \in PSL(2,\mathbb{R})$ s.t. $A(\phi(v_0)) = v_0$.

Then $A \circ \phi$ is an isometry fixing a tangent vector,

hence $A \circ \phi = Id_{\mathbb{H}^2}$ by some argument above.

Thus $\phi = A^{-1}$, & map is surjective ☒

Classification of Isometries of $\mathbb{H}^2$

Every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \cong Isom^+(\mathbb{H}^2)$ is of 1 of 3 types:

Note: A gives homeom $\underbrace{\mathbb{H}^2 \cup \mathbb{R}^2}_{\text{closed disk}} \to \mathbb{H}^2 \cup \hat{\mathbb{R}}$

$\overset{\text{Brouwer}}{\Longrightarrow}$ A has at least 1 fixed pt in $\mathbb{H}^2 \cup \hat{\mathbb{R}}$!

$$\boxed{\left.\begin{array}{l} \bullet\ 2 \text{ fixed pts in } \mathbb{H}^2 \\ \quad \text{or} \\ \bullet\ 1 \text{ fixed pt in } \mathbb{H}^2 \text{ \& } 1 \text{ in } \hat{\mathbb{R}} \end{array}\right\} \Rightarrow A = \text{Identity (must fix a tangent vector)}}$$

for $x \in \mathbb{R}$, eqn $\frac{ax+b}{cx+d} = x$; $cx^2 + (d-a)x - b$

roots: $x = \dfrac{a-d \pm \sqrt{(d-a)^2 + 4bc}}{2c}$

disc: $(d-a)^2 + 4bc = a^2 - 2ad + d^2 + 4bc$

$= a^2 + 2ad + d^2 - 4ad + 4bc$

$= \boxed{(a+d)^2 - 4}$

$\Rightarrow \frac{ax+b}{cx+d} = x$ has $\begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$ soln iff $a+d \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is:

1) <u>Elliptic</u> if $tr(A) = a+d < 2$

- no fixed pts in $\hat{\mathbb{R}}$
- A fixes exactly one pt in $\mathbb{H}^2$; rotation about that pt.
- conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ some $\theta \in \mathbb{R}$ (fixes i)

2) <u>Parabolic</u> if $tr(A) = a+d = 2$.

- Exactly 1 fixed pt in $\hat{\mathbb{R}}$
- no fixed pts in $\mathbb{H}^2$
- conjugate to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ some $t \in \mathbb{R}$ (fixes ∞)

3) <u>Hyperbolic</u> if $tr(A) = a+d > 2$

- Exactly 2 fixed pts in $\hat{\mathbb{R}}$
- no fixed pts in $\mathbb{H}^2$
- conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ some $\lambda > 0$ (fixes 0 & ∞)

Talk about how you know conj to these!

Say $\varphi \in \text{Isom}^+(\mathbb{H}^2)$ is hyperbolic, w/ fixed pts $x, y \in \hat{\mathbb{R}}$

Let $\alpha : \mathbb{R} \to \mathbb{H}^2$ be geodesic from x to y.

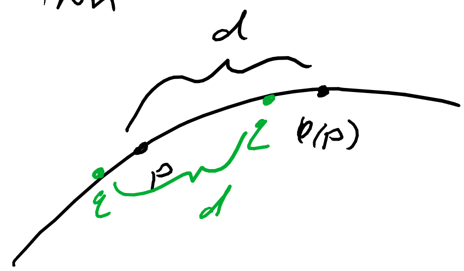
param by arclength: $\ell_{\mathbb{H}^2}(\alpha|_{[a,b]}) = |a-d|$

Then: $\varphi(\alpha) = \alpha$, so φ restricts to isometry of $\mathbb{R}$:

must give <u>translation</u> along α!

Pick $t_1, t_2 \in \mathbb{R}$, say $t_1 < t_2$

write $\varphi(\alpha(t_i)) = \alpha(t_i + d_i)$, $d_i \in \mathbb{R}$. Then



$$\ell(\alpha|_{[t_1,t_2]}) = \ell(\varphi(\alpha|_{[t_1,t_2]}))$$

$$\begin{array}{ccc} \| & & \| \\ t_2 - t_1 & & (t_2 + d_2) - (t_1 + d_1) \end{array}$$

$= \boxed{d_2 = d_1}$ so every pt translates by dist $d = d_1 = d_2$!

d is called the translation length of φ. $\boxed{\tau(\varphi)}$

α called the axis of φ.

If φ translates from x to y in $\hat{\mathbb{R}}$,

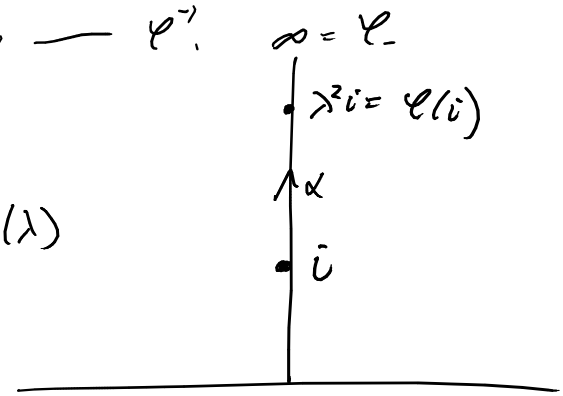
call $x = \varphi_-$ reppelling fixed pt

$y = \varphi_+$ attracting fixed pt

Notice: in $\mathbb{H}^2 \cup \hat{\mathbb{R}}$ (cpct), everthing converges

towards φ_+ under iteration of φ

— " — φ_- ——— " ——— φ^{-1}.



Ex $\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$, $\lambda > 1$

length path i to $\lambda^2 i$ is $\ln(\lambda^2/1) = 2\ln(\lambda)$

Translation length is:

$$\boxed{\tau = \tau\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = |2\ln(\lambda)|}$$

End 1/26/18

$\tau = 2\ln(\lambda)$, $\lambda = e^{\tau/2}$, so get

$\begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix}$, trace $T = e^{\tau/2} + e^{-\tau/2}$

$$\boxed{\text{trace} = 2\cosh(\tau/2)}$$

$$\boxed{\tau = 2\,\mathrm{arccosh}(\text{trace}/2)}$$

$$= 2\ln\left(\frac{Tr + \sqrt{Tr^2 - 4}}{2}\right)$$

Δ trans length is conj. invt in $PSL(2,\mathbb{R})$!!

Mon 1/29/18 Conv: defn trans len(elliptic/parabolic) $= 0$!

Hyperbolic Surfaces

G group, X top space. Recall that a group action $G \curvearrowright X$ is:

- properly discontinuous if $\forall K \subset X$ compact, the set
$$\{g \in G \mid g\cdot K \cap K \neq \emptyset\} \text{ is finite}$$
- free if $\forall x \in X$, stabilizer $G_x = \{g \in G \mid g\cdot x = x\}$ is trivial.

Fact: If $G \curvearrowright X$ freely + properly discontinuously, + X Hausdorff, then the quotient map $X \longrightarrow X/G = \{G\cdot x \mid x \in X\}$, w/ $\underbrace{\text{quotient top}}_{\text{subset of } X/G \text{ open iff preimage open.}}$ is a covering map.

(exercise if you have not seen/thought about in a while.

Consequence If $\Gamma \leq PSL(2,\mathbb{R})$ subgroup s.t. $\Gamma \curvearrowright \mathbb{H}^2$ freely & prop disc. then $\pi: \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ is covering map.

$\rightsquigarrow$ $\mathbb{H}^2/\Gamma$ inherits smooth structure st π smooth

$Deck(\pi) = \Gamma \leq Isom^+(\mathbb{H}^2) \implies \mathbb{H}^2/\Gamma$ inherits Riem. metric st π local isometry.

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3 definitions of a hyperbolic surface:

Defn 1 A oriented hyperbolic surface is the quotient $\mathbb{H}^2/\Gamma$ of $\mathbb{H}^2$ by a subgroup $\Gamma \leq PSL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$, equipped with the quotient topology + smooth structure + Riem metric st $\mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ is is local isometry.

Defn 2 An (oriented) hyperbolic surface is an oriented 2-dim'l smooth mnfld M equipped w/ Riem mnfld st universal cover $\tilde{M}$ is isometric to $\mathbb{H}^2$.

Note:
- choosing isometry $\tilde{M} \to \mathbb{H}^2$ induces monomorphism $\mathrm{Deck}(\tilde{M} \to M) \to \mathrm{Isom}^+(\mathbb{H}^2)$
- Choosing basepts $p \in M$ + lift $\tilde{p} \in \tilde{M}$ induces isomorphism $\pi_1(M,p) \cong \mathrm{Deck}(\tilde{M})$ $(\leq PSL(2,\mathbb{R}))$

$\rightsquigarrow$ get isomorphism $\pi_1(M,p) \longleftrightarrow$ subgrp $\Gamma \leq PSL(2,\mathbb{R})$ acting freely + prop. disc. on $\mathbb{H}^2$

- Changing lift $\tilde{p}$ changes $\pi_1(M,p) \xrightarrow{\cong} \mathrm{Deck}(\tilde{M})$ by post conjugation by an elt. of Deck

- Changing basept p also changes $\pi_1(M,p) \to \mathrm{Deck}(\tilde{M})$ by conj.

- Changing the isometry $\tilde{M} \to \mathbb{H}^2$ changes $\mathrm{Deck}(\tilde{M}) \cong PSL(2,\mathbb{R})$ by conjugation by some elt. of $PSL(2,\mathbb{R})$.

Upshot: for any hyp surf M, get injection

$$\pi_1(M,p) \longrightarrow PSL(2,\mathbb{R}) \text{ well-defined up to post conjugation in } PSL(2,\mathbb{R})$$

Defn 3 A hyp. surface is a 2-dim smooth manifold M together with an elt $\phi \in \mathrm{Hom}(\pi_1(M,p), PSL(2,\mathbb{R})) \Big/ \underbrace{PSL(2,\mathbb{R})}_{\text{conj}}$

that is

- discrete (Image(ϕ) is discrete subgrp $PSL(2,\mathbb{R})$) (w/ disc. topology)
  $\Rightarrow$ Img. acts prop. disc. & freely.
- &
- faithful ($\ker(\phi)$ trivial)
  $\Rightarrow$ Image $\cong \pi_1(M)$

↳ Gives us a hint of how one may parametrize space of hyperbolic structures on fixed top surface!

Curves & lengths

Let X be a top space.

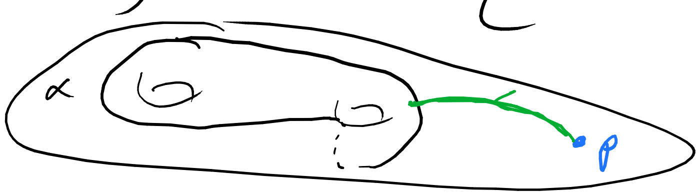
A curve in X is a cont. map $\alpha: S^1 \to X$

curves α & β are freely homotopic iff the maps $\alpha: S^1 \to X$ & $\beta: S^1 \to X$ are homotopic.

— generally only consider curves up to free homotopy.

Recall: for any basept $p \in X$, get bijection

$$\left\{\begin{array}{c}\text{free homotopy}\\ \text{classes of}\\ \text{curves in } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{conjugacy classes}\\ \text{in } \pi_1(X,p)\end{array}\right\}$$



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If M hyp surf, $\forall p \in M$ get monomorphism $\pi_1(M, p) \to PSL(2,\mathbb{R})$, well-defined up to conj in $PSL(2,\mathbb{R})$

Upshot:

In a hyp surface, each free homotopy class of curves corresponds to a well-defined conj class in $PSL(2,\mathbb{R})$

$\leadsto$ Define length of a free homotopy class of curves on M to be translation length of corresp. conj class in $PSL(2,\mathbb{R})$.
(well-defined!)