Math 9201 – Seminar in Topology – Spring 2018

Seminar in Topology – Simple curves and the Volumes of Moduli Spaces: An exploration of the Thesis of Maryam Mirzakhani

MWF 1:10–2:00 pm
SC 1117

Mon 1/8/18

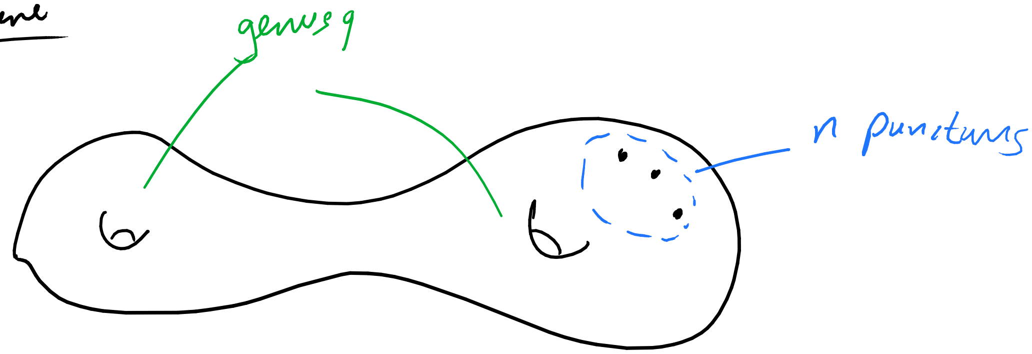
- Introductions
- Go over Syllabus, highlight:
  - plan, texts, prerequisites, grades, presentations, Outline
- Hope we can have interactive atmosphere, feel free to ask questions, slow me down, etc.
- Discuss end of term:
  - I will be gone last 2 weeks (April 9–20), last day is April 23 for some of these days we will have a sub or student presentations, other days class may be cancelled.
  - To make up for the missed classes, we will have:

Make up days   Wed April 25
                Fri April 27

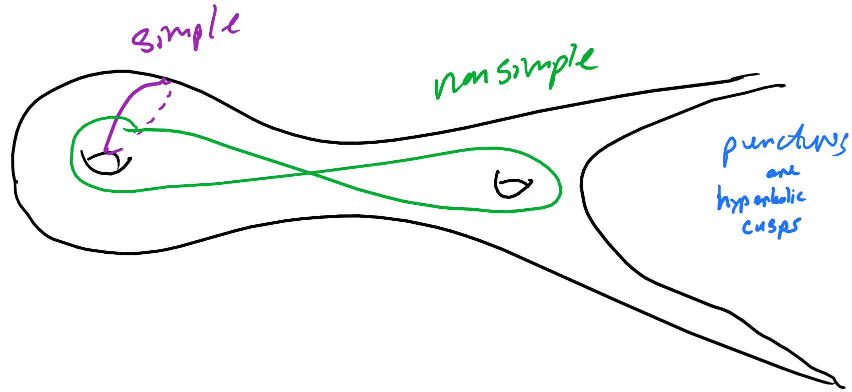
(Distribute forms & get them to agree to extra lectures during exam week)

Broad Goal / Outline

top surface S



hyp surf X = top surface equipped w/ hyperbolic Riem metric



(1) Study X itself

curves, geodesics, length spectrum, simple curves

★ growth rate of # of simple closed curves!

each curve $\alpha : S^1 \to X$ has a unique geod. in its homotopy class. has a length $\ell_X(\alpha)$

curve is simple if no self intersections.

$$S_X(L) = \left\{ \begin{matrix} \text{geodesics} \\ \alpha \end{matrix} \;\middle|\; \begin{matrix} \alpha \text{ simple } + \\ \ell_\alpha(X) \leq L \end{matrix} \right\}$$

How does it grow with L?

(Mirzakhani) $S_X(L) \asymp n_X L^{6g-6}$

Further: Mirzakhani gives info about how n_X depends on X.

Contrast: $C_X(L) = \left\{ \begin{matrix} \text{geodesics} \\ \alpha \text{ on } X \\ \text{non simple} \end{matrix} \;\middle|\; \ell_X(\alpha) \leq L \right\}$, (Margulis) $C_X(L) \asymp \dfrac{e^L}{L}$

Study moduli space $\mathcal{M}(S)$ of possible hyperbolic metrics on surface S.

- natural topology (non compact)

  coming from: length functions nice coordinates (Fenchel-Nielsen)

  look at: · homology / cohomology
  · fundamental group
  · universal cover
  · compact part
  · noncompact part: ends, structure, boundary at infinity

- several natural metrics:

  Teichmuller, Weil-Petersson, Thurston / Lipschitz.

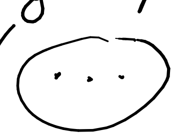
  Riemannian

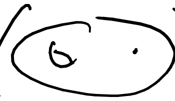
  look at: · distances
  · geodesics,
  · geometry / curvature
  · volumes — we focus on this!

Let $\mathrm{Vol}_{wp}(S)$ be volume of moduli space $\mathcal{M}(S)$ equipped with Weil Petersson volume form

(Mirzakhani) Gives explicit recursive formula expressing $\mathrm{Vol}_{wp}(S)$ in terms of volumes $\mathrm{Vol}_{wp}(S')$ of smaller complexity surfaces S'

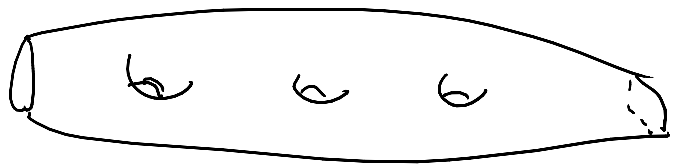
- lets you explicitly compute volume (in low complexity)

eg: $\mathrm{Vol}_{wp}($  $) = 1$

$\mathrm{Vol}_{wp}($  $) = \pi^2/6$

$\mathrm{Vol}_{wp}($  $) = \pi^4/8$

instead of punctures, can also look at surfaces with boundary.



study $\mathcal{M}_{g,n}(l) = \mathcal{M}_{g,n}(l_1, \ldots, l_n)$

= moduli space of hyp metrics on genus g surf. with n totally geodesic boundary components of prescribed lengths $l_1, \ldots, l_n$.

— still get nice moduli space w/ coordinates, WP metric & volume.

now look at $V_{g,n}(l)$ = Weil-Petersson volume of $\mathcal{M}_{g,n}(l)$.

Turns out: Fixing g, n, $V_{g,n}(l)$ is polynomial in $l_1, \ldots, l_n$

(Mirzakhani) — calculate these recursively as above

- coefficients are related to intersection pairings between certain tautological bundles over $\mathcal{M}_{g,n}(l)$.

— These coefficients/intersection pairings are shown to satisfy Virasoro Equations & this leads her to a new proof of Witten-Kontsevich Thm.

(This part of the story I knew less about and am less interested in, so we will not focus on it as much, but will see how it goes.)

I. Background

I assume basic topology: topological spaces, fund. group, covering spaces, compactness, etc.

For open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, a map $\varphi : U \to V$ is <u>smooth</u> if all partial derivatives (of all orders) exist (& hence are continuous) everywhere in U.

A <u>smooth atlas</u> on a topological space X is a collection $\{\varphi_j : U_j \to V_j\}$ of "charts", where

- $\{U_j\}$ is an open cover of X
- V_j open in $\mathbb{R}^n$
- $\varphi_j : U_j \to V_j$ a homeomorphism
- "transition functions" $T_{ij} = \varphi_j \circ \varphi_i^{-1}\Big|_{\varphi_i(U_i \cap U_j)}$ are smooth (when defined).

- 2 smooth atlases are equivalent if their union is a smooth atlas.

<u>Def</u> An n-dimensional <u>smooth manifold</u> is a (second countable, Hausdorff) top space M equipped with an equivalence class of smooth atlases.

<u>Ex</u> $M = \mathbb{R}^n$ with atlas $\{id : \mathbb{R}^n \to \mathbb{R}^n\}$

<u>Def</u> A map $f : M \to N$ of smooth manifolds is <u>smooth</u> if $\psi \circ f \circ \varphi^{-1}$ is smooth (when defined) for all charts $\varphi : U \to V$ on M & $\psi : W \to Z$ on N.

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A <u>diffeomorphism</u> is a smooth bijection w/ smooth inverse.

M smooth n-mnfld. $p \in M$

$C_p^\infty(M)$ = set of equivalence class of smooth functions $f: U \to \mathbb{R}$, where $U \subset M$ open & $p \in U$.

where $g: V \to \mathbb{R}$, $f: U \to \mathbb{R}$ are equivalent if $\exists$ open set $Y \subset V \cap U$ with $p \in Y$ & $f|_Y = g|_Y$

- $C_p^\infty(M)$ is an $\mathbb{R}$-vector space.

A <u>derivation at p</u> is a mapping $D: C_p^\infty(M) \to \mathbb{R}$ satisfying:

- (linear) $D(\lambda f + \mu g) = \lambda D(f) + \mu D(g)$

(Leibniz rule) $D(fg) = D(f)g(p) + f(p)D(g)$.

$T_p(M)$ = set of derivations at p.

- naturally an $\mathbb{R}$ vector space.

☆ a derivation is a <u>directional</u> <u>derivative</u>

<u>Ex 1)</u> Let $\gamma: (-\varepsilon, \varepsilon) \to M$ be smooth with $\gamma(0) = p$.

define $\frac{\partial \gamma}{\partial t}: C_p^\infty(M) \to \mathbb{R}$ by: (also write $\gamma'(0)$ or $\left.\frac{\partial \gamma}{\partial t}\right|_{t=0}$)

for $f \in C_p^\infty(M)$, get $f \circ \gamma: (-\delta, \delta) \to \mathbb{R}$ well defined for all suff small $\delta > 0$

set $\frac{\partial \gamma}{\partial t}(f) = \left.\frac{d}{dt}\left(f \circ \gamma(t)\right)\right|_{t=0}$.

- This is a derivation.
- Every derivation arises in this way!

Ex 2 Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart with $p \in U$.

For $i = 1, \dots, n$, define

$$\left(\frac{\partial}{\partial x_i}\right)_p : C_p^\infty(M) \to \mathbb{R} \text{ by}$$

$$\left(\frac{\partial}{\partial x_i}\right)_p (f) = i^{th} \text{ partial derivative of } f \circ \varphi^{-1} \text{ at } \varphi(p).$$

Exercise: $\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p \right\}$ is a basis of $D_p(M)$.

Def The tangent space of M at p is the n-dimensional vector space $T_pM =$ of derivations at p.

↳ elements called tangent vectors

The tangent bundle of M is

$$TM = \{ v \mid v \in T_pM \text{ for some } p \in M \}$$

projection map $\pi : TM \to M$, $\pi(v) = p$, where $v \in T_pM$.

TM is a smooth manifold: If $\{ \varphi_j : U_j \to V_j \}$ is a smooth atlas of M, then $\{ \psi_j : \pi^{-1}(U_j) \to V_j \times \mathbb{R}^n \}$ is a smooth atlas, where $\psi_j^{-1} : V_j \times \mathbb{R}^n \to \pi^{-1}(U_j)$ is

$$\psi_j^{-1}(q, a_1, \dots, a_n) = \sum_{i=1}^{n} a_i \left(\frac{\partial}{\partial x_i}\right)_{\varphi_j^{-1}(q)} \in T_{\varphi_j^{-1}(q)} M$$

Exercise: 1) Define appropriate topology on TM

2) Verify this gives smooth atlas on TM

3) $\pi : TM \to M$ is smooth

Derivatives Let $f: M \to N$ be a smooth map of smooth manifolds. For each $p \in M$, the derivative of f at p is the linear map $D_p f : T_p M \to T_{f(p)} N$ given by:

for $v : C_p^\infty(M) \to \mathbb{R}$ a derivation at p,

$D_p f(v) : C^\infty_{f(p)}(N) \to \mathbb{R}$ is the derivation

$$g \longmapsto v(g \circ f)$$

Note $g \circ f \in C_p^\infty(M)$

Check: · $D_p f(v)$ is indeed a derivation at $f(p)$

· $D_p f$ is a linear map

Ex $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, $(D_p f)\left(\frac{\partial \gamma}{\partial t}\right)_p = \left(\frac{\partial (f \circ \gamma)}{\partial t}\right)_{f(p)}$

Putting together for all $p \in M$, get derivative map

$Df : TM \to TN$, $Df(v) = D_p f(v) \in T_{f(p)} N$, where $v \in T_p M$

— smooth.

Vector Bundles

Let B be a topological space.

Def a $\mathbb{R}$-vector bundle ξ over B consists of:

1) a topological space $E = E(\xi)$, "total space"

2) continuous map $\pi : E \to B$ "projection map"

3) For each $b \in B$, an $\mathbb{R}$-vector space structure on the set $\pi^{-1}(b)$ "fiber over b" — Denoted E_b

Satisfying local triviality condition: for each $b \in B$ $\exists$ nbhd $U \subset B$, integer $n \geq 0$, and homeo $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ s.t. $\forall u \in U$, the map $x \mapsto h(u, x)$ gives a vect space isomorphism $\mathbb{R}^n \cong \pi^{-1}(u)$.

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* note: implies n constant on each component of B } n-plane bundle.

~~Fri 1/12/18~~ Snow Day Wed 1/17/18

A smooth vector bundle defined similarly:
require E, B to be smooth manifolds, π to be smooth,
and local trivializations $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ to be diffeomorphisms

Ex For M any n-dim'l smooth mfld, the tangent bundle
$\pi : TM \to M$ is a smooth vector bundle.
(Indeed: the charts we defined give local trivializations)

Def a (smooth) <u>section</u> of a vect bundle $\{ \pi : E \to B$,
is a cont (smooth) map $\sigma : B \to E$ s.t.

$$\sigma(b) \in \pi^{-1}(b) \quad \forall b \quad (\text{i.e. } \pi \circ \sigma = Id_B)$$

<u>nowhere zero</u> if $\forall b \in B$, $\sigma(b) \underbrace{\neq 0 \in E_b}_{\text{zero vect in the vect space}}$

Def A <u>vector field</u> X on M, is a "smooth" choice
of a tangent vector $X_p \in T_pM$ for each $p \in M$.
More precisely, a vector field is a smooth section of $TM \to M$.

- If X a vector field + $f : M \to \mathbb{R}$ smooth function, then
$X(f) : M \to \mathbb{R}$, $X(f)(p) = X_p(f)$ is smooth funct. on M
"derivative of f in dir X"

Operations on Vector bundles

Any operations / constructions of vector space
can be applied to vector bundles over a fixed base
space B: just perform the operation <u>fiberwise</u>.

<u>Ex</u> Suppose $\xi \ \pi: E \to B$ & $\xi' \ \pi': E' \to B$

are 2 vector bundles over B.

1) The <u>direct sum</u> or Whitney sum of ξ & ξ' is

bundle $\xi \oplus \xi'$ where fiber over $b \in B$ is

direct sum $E_b \oplus E_b'$

(so total space is

$$\{(v,v') \in E \times E' \mid \pi(v) = \pi'(v')\}$$

- has natural map to B.
- fibers are clearly $E_b \times E_b'$; give Cartesian product vector space structure
- Think about how to define the local trivializations

② <u>Tensor product</u> $\xi \otimes \xi'$; fiber over $b \in B$ is

$$E_b \otimes E_b'$$

- Think about local trivialization: build it from local trivializations of ξ and ξ'

②' Tensor powers: $\xi^{\otimes k}$ apply above iteratively

③ <u>Dual</u> ξ^*: vector bundle whose fiber over $b \in B$ is the dual vector space $(E_b)^*$

(again, the local trivializations are easy to define; eg use a basis)

④ <u>Exterior powers</u>: $\xi^{\wedge k}$: fibers are $\Lambda^k E_b$.

M smooth manifold, $p \in M$ write

$$T_p^* M = \text{dual vector space of } T_p M$$
$$= \text{set of all maps } T_p M \to \mathbb{R}$$

- "cotangent space"
- a <u>cotangent vector</u> (at p) is an element of $T_p^* M$.

The <u>cotangent bundle</u> is the bundle $T^* M \to M$ whose fiber over $p \in M$ is $T_p^* M$.

↳ The dual of the vector bundle $TM \to M$.

<u>Def</u> A 1-form on M is a (smooth) section of $T^* M \to M$ (i.e. smooth choice of cotangent vector $\sigma_p \in T_p^* M$ for each $p \in M$).

If X a vector field on M & σ a 1-form, then

$$\sigma(X): M \to \mathbb{R} \quad \text{is smooth}$$
$$p \mapsto \sigma_p(X_p)$$

<u>Ex</u>: Exterior derivative:

Each smooth function $f: M \to \mathbb{R}$ determines a 1-form df — the <u>differential</u> of f

given $p \in M$, $df_p \in T_p^* M = \mathrm{Hom}(T_p M, \mathbb{R})$

is defined by: $\underset{\substack{\cap \\ T_p M}}{v} \mapsto v(f)$ — clearly linear.

- if f only defined in an open set $U \subset M$, then df_p makes sense for each $p \in U$.

Ex Let $\varphi: U \to V \subset \mathbb{R}^n$ be chart

for $i = 1, \dots, n$, let $x_i = (\text{proj onto } i\text{th coord}) \circ \varphi : U \to \mathbb{R}$

so $x_1, \dots, x_n$ are smooth functions on U.

get sections $dx_1, \dots, dx_n : U \to T^*M$ (1-forms defined on U)

Claim: for $p \in U$, $(dx_1)_p, \dots, (dx_n)_p$ is a basis of T_p^*M.

In fact it is the dual to the basis $\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p \right\}$ of T_pM.

proof: fix i, j.

$$(dx_j)_p \left(\frac{\partial}{\partial x_i}\right)_p = \left(\frac{\partial}{\partial x_i}\right)_p (x_j)$$

$$= \begin{array}{c}\text{ith partial derivative} \\ \text{at } q = \varphi(p) \text{ of}\end{array} \left(V \xrightarrow{\varphi^{-1}} U \xrightarrow{\varphi} V \xrightarrow{j\text{th proj}} \mathbb{R} \right)$$

$$= \begin{array}{c}\text{ith partial} \\ \text{derivative at } q = \varphi(p)\end{array} \left(j\text{th proj } V \to \mathbb{R} \right)$$

$$= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\blacksquare$

A <u>Riemannian metric</u> g on M is a smooth choice of an inner product $g_p(\cdot,\cdot)$ on T_pM for each $p \in M$.

That is: for each $p \in M$, have a pairing

$$g_p : T_pM \times T_pM \to \mathbb{R} \quad \text{that is}$$

1) bilinear (linear in each entry)

2) symmetric $\left(g_p(v,u) = g_p(u,v) \right)$

3) positive definite $\left(g_p(v,v) \geq 0 \ \forall v \in T_pM \ \& \ g_p(v,v) = 0 \text{ iff } v = 0 \right)$

smooth means that for any vector fields X, Y on M,

$$g(X,Y) : M \to \mathbb{R} \quad \text{is smooth.}$$
$$p \mapsto g_p(X_p, Y_p)$$

<u>Exercise</u>: a Riem. metric g defines a section of $T^*M \otimes T^*M \to M$ (satisfying extra conditions)

<u>Example</u> $M = \mathbb{R}^n$. Have single chart $id : \mathbb{R}^n \to \mathbb{R}^n$

$\leadsto$ at any $p \in M$, partial derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ give basis of $T_p\mathbb{R}^n$

define pairing $g_p : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$

$$g_p\left(\sum a_i \left(\frac{\partial}{\partial x_i}\right)_p, \sum b_i \left(\frac{\partial}{\partial x_i}\right)_p \right) = \sum a_i b_i = a_1 b_1 + \dots + a_n b_n$$

(std inner product)

Then g is a Riemannian metric on $\mathbb{R}^n$!

A map $f: (M,g) \to (N,h)$ of Riem. mnflds is a <u>local isometry</u> if for each $p \in M$ the derivative $D_pf: T_pM \to T_{f(p)}N$ is an isometry of the inner products g_p & $h_{f(p)}$. That is: D_pf isomorphism and

$$g_p(v,u) = h_{f(p)}(D_pf(v), D_pf(u)) \quad \forall v,u \in T_pM.$$

D_pf injective

· <u>PullBack</u> If N has Riem metric h, and $f: M \to N$ local differ (By inv. funct. Thm: equiv D_pf isom $\forall p \in M$) then $\exists!$ Riem metric $f^*(h)$, <u>pull back</u> s.t. f local isometry: at $p \in M$ set

$$\forall v,u \in T_pM: \quad f^*(h)_p(v,u) = h_{f(p)}(D_pf(v), D_pf(u))$$

<u>Def</u> an <u>isometry</u> of Riem. mnflds is a diffeomorphism that is also a local isometry

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(equiv: a local isometry that is bijective).

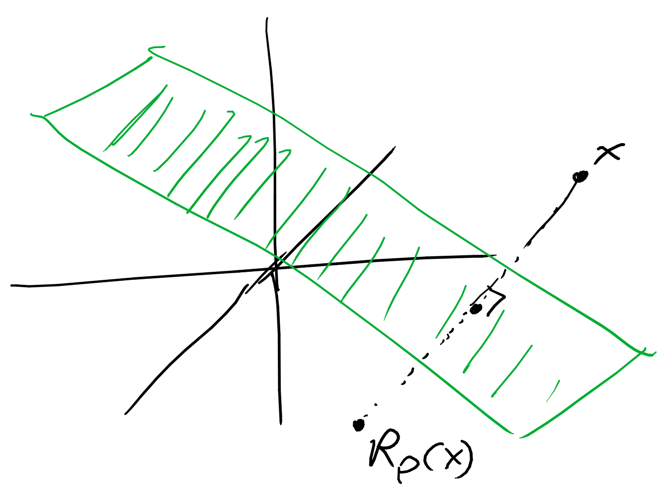
The <u>isometry group</u> of M is $Isom(M) = \{ f: M \to M \mid f \text{ an isometry} \}$

<u>Example</u> Isometries of $\mathbb{R}^n$ (w/ std metric):

· <u>translations</u>: for $v \in \mathbb{R}^n$ fixed, $A_v: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x+v$

$\rightsquigarrow$ injective group homom $\mathbb{R}^n \to Isom(\mathbb{R}^n)$, $v \mapsto A_v$. Image $A \cong \mathbb{R}^n$

· <u>Reflection</u> R_P in a plane $P \subset \mathbb{R}^n$



<u>Rmk</u>: $Isom(\mathbb{R}^n)$ generated by reflections

General linear group

$$GL(n,\mathbb{R}) = \left\{ A \ \begin{matrix} n\times n \text{ real} \\ \text{matrix} \end{matrix} \ \middle| \ \det(A) \neq 0 \right\}.$$

(group under matrix multiplication).

$GL(n,\mathbb{R}) \curvearrowright \mathbb{R}^n$ by linear maps:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in GL(n,\mathbb{R}), \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$A \cdot x = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

get injective group homom $GL(n,\mathbb{R})$

Thus $A \in GL(n,\mathbb{R}) \rightsquigarrow$ $\underline{\text{diffeo}}$ $A \cdot \mathbb{R}^n \to \mathbb{R}^n$

gives injective homomorphism

$$GL(n,\mathbb{R}) \to \mathrm{Diffeo}(\mathbb{R}^n)$$

$\underline{\text{Exercise}}$: this diffeo $A: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ (w/ std metric)

$$\Longleftrightarrow$$

identity matrix

$$A \in O(n) = \{ A \in GL(n,\mathbb{R}) \mid A^T A = I_n \}$$

$\underline{\text{orthogonal group}}$

Now have 2 subgroups of $\mathrm{Isom}(\mathbb{R}^n)$:

$\mathbb{R}^n$ = group of translations

$O(n)$ = orthogonal subgroup of $GL(n,\mathbb{R}) \leq \mathrm{Diffeo}(\mathbb{R}^n)$

$\underline{\text{Fact}}$ (exercise) these give a semidirect product decomp:

$$\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$$

use fact that isometry determined by derivative at one pt!

G grp. $A, B \leq G$. $G = A \rtimes B$ means:

1) $A \trianglelefteq G$ 2) $\{ ab \mid a \in A, b \in B \} = G$ 3) $A \cap B = \{e\}$

Covers

- a cont. map $p: Y \to X$ of top spaces is a **covering map** if:

  $\forall x \in X$ $\exists$ nbhd $U \subset X$ of x s.t.

  $p^{-1}(U) = \{y \in Y \mid p(y) \in U\}$ is a disjoint union of open sets, each of which maps homeomorphically onto U by p — U is "evenly covered"

- The **deck group** of the cover is

$$\mathrm{Deck}(Y \xrightarrow{p} X) = \{f: Y \to Y \text{ homeo} \mid p \circ f = p\} \leq \mathrm{Homeo}(Y)$$

Facts: Suppose $p: Y \to X$ covering map

1) If X smooth mnfld, $\exists!$ smooth structure on Y s.t. p local diffeo

   Take atlas $\{\varphi_i : U_i \to V_i\}$ of X s.t. each U_i evenly covered.

   Then charts $\psi: W \to V$,

   W = cmpt of $p^{-1}(U_i)$ some i +

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[rr, bend left, "\psi"] & U_i \arrow[r, "\varphi_i"] & V_i \end{tikzcd}$$

   give smooth atlas.

   !! may be other smooth structures on Y s.t. p smooth! use smooth homeo $Y \to Y$ that is not diffeo.

$\Rightarrow$ If X has Riem metric g get pull back Riem metric $p^*(g)$ on this smooth mnfld Y (so p local isom).

2) If Y smooth mnfld and each $f \in \mathrm{Deck}(Y \xrightarrow{p} X)$ is smooth, get unique smooth str on X s.t. p local diffeo

   Choose cover $\{U_\alpha\}$ of X s.t. each U_α evenly covered + some cpt W_α of $p^{-1}(U_\alpha)$ contained in some chart $\psi: U \to V$ of Y.

   def $\varphi_\alpha = \psi \circ p^{-1} : U_\alpha \to W_\alpha \to \psi(W_\alpha) \subset \mathbb{R}^n$.

   check: gives smooth atlas (use fact that deck transf. smooth)

2') If Y smooth w/ Riem metric g s.t. each $f \in \mathrm{Deck}(Y \xrightarrow{p} X)$ is an isometry, then g descends to unique Riem metric $p_*(g)$ on X s.t. p local isometry:

for $v, u \in T_x X$, choose $y \in p^{-1}(x)$ & set

$$p_*(g)_x (v,u) = g_y(\tilde{v}, \tilde{u}), \text{ where } D_y p(\tilde{v}) = v, \; D_y p(\tilde{u}) = u.$$

(well defined since each deck transf. is isometry.)

Curves

a (piecewise smooth) path in M is a map $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an interval that is smooth (except at finitely many points)

Derivative $\quad \gamma'(t) = D_t \gamma\left(\frac{d}{dx}\right) \in T_{\gamma(t)} M \quad = \text{derivation } f \mapsto \frac{d(f \circ \gamma)}{dt}(t)$

similarly: (piecewise smooth) closed curve is map $\gamma : S^1 \to M$. (circle)

Length Let M be Riem. mnfld with metric g.

for a p.w smooth path $\gamma : [a,b] \to M$

length of γ is

$$\ell_M(\gamma) = \int_a^b \left\| \frac{\partial \gamma}{\partial t} \right\| dt = \int_a^b \underbrace{\|\gamma'(t)\|}_{} dt,$$

$$\sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} =$$

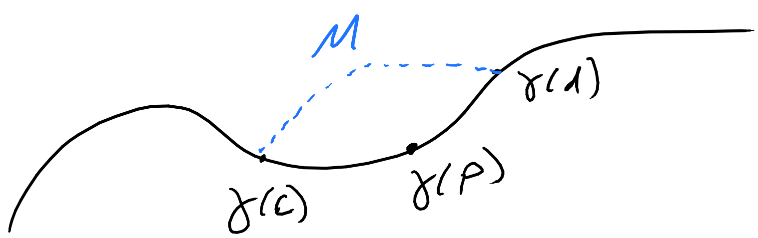
similarly define length of a (piecewise) smooth closed curve: $\gamma : S^1 \to M$

Ex For $\mathbb{R}^3$ with std Riem metric, this is usual notion of path length we all learn in calculus

Def A geodesic in a Riem mnfld M is a path $\gamma : [a,b] \to M$ or closed curve $\gamma : S^1 \to M$ that is locally length minimizing.

That is: $\forall p \in (a,b)$ $\exists$ $a \le c < p < d \le b$ s.t.

$$l_M(\gamma|_{[c,d]}) \le l_M(\mu) \text{ for any path } \mu \text{ from } \gamma(c) \text{ to } \gamma(d)$$



can always (re)parametrize geodesic by arclength so that:

$$l_M(\gamma|_{[c,d]}) = |d-c| \quad \text{any subinterval } [c,d] \subset [a,b]$$

$\iff$ unit speed: $\|\gamma'(t)\| = 1 \quad \forall t$

Rmk ① Usually geodesics defined to be paths satisfying a differential equation: "accelleration = 0"
(requires having notion of connection, which we don't need)
But that analytic defn is equivalent to this

② consequence of existence/uniqueness of solutions to ODEs:

Thm Given $p \in M$, $\exists$ nbhd $V \subset M$ of p, $\exists\, \delta, \varepsilon > 0$ +
smooth map $\gamma : (-\delta, \delta) \times V_\varepsilon \to M$,
where $V_\varepsilon = \{ v \in TM \mid \pi(v) \in V,\ \|v\| < \varepsilon \}$, s.t.
$\forall v \in V_\varepsilon$, curve $t \mapsto \alpha(t) = \gamma(t, v)$ is unique
const-speed geod. with $\alpha(0) = \pi(v)$ & $\gamma'(0) = v$.

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$\hookrightarrow$ locally, $\exists!$ geodesic through any tangent vector
- can then try to extend this geod longer & longer
- If every geod may be extended indefinitely, the Riem mnfld is said to be complete