Math 7210: Riemannian Geometry – Homework 2

Due in class: Wednesday, January 25, 2017

1. (from #0.7 of do Carmo) Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ be a right circular cylinder, and let $A\colon C \to C$ be the symmetry with respect to the origin, that is $A(x, y, z) = (-x, -y, -z)$. Let M be the quotient space of C with respect to the equivalence relation $p \sim A(p)$, and let $\pi\colon C \to M$ be the projection $\pi(p) = \{p, A(p)\}$.
   (a) Show that 1 is a regular value of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2$; meaning that the derivative d_pf is surjective for all $p \in f^{-1}(1)$. It thus follows from the submersion theorem that $C = f^{-1}(1)$ is a submanifold of $\mathbb{R}^3$ of dimension 2.
   (b) Show that it is possible to give M a differentiable structure such that π is a local diffeomorphism.
   (c) Prove that M is non-orientable.

2. (from #0.8 of do Carmo) Let M and N be differentiable manifolds and let $\varphi\colon M \to N$ be a local diffeomorphism. Prove that if N is orientable, then M is orientable.

3. (from #0.11 of do Carmo) Let us consider the two following differentiable structures on the real line $\mathbb{R}$: $(\mathbb{R}, \varphi_1)$, where $\varphi_1\colon \mathbb{R} \to \mathbb{R}$ is given by $\varphi_1(x) = x$ for $x \in \mathbb{R}$; $(\mathbb{R}, \varphi_2)$, where $\varphi_2\colon \mathbb{R} \to \mathbb{R}$ is given by $\varphi_2(x) = x^3$ for $x \in \mathbb{R}$. Show that
   (a) the identity mapping $i\colon (\mathbb{R}, \varphi_1) \to (\mathbb{R}, \varphi_2)$ is not a diffeomorphism; therefore the maximal structures determined by $(\mathbb{R}, \varphi_1)$ and $(\mathbb{R}, \varphi_2)$ are distinct.
   (b) the mapping $f\colon (\mathbb{R}, \varphi_1) \to (\mathbb{R}, \varphi_2)$ given by $f(x) = x^3$ is a diffeomorphism; that is, even though the differentiable structures $(\mathbb{R}, \varphi_1)$ and $(\mathbb{R}, \varphi_2)$ are distinct, they determine diffeomorphic differentiable manifolds.

4. Let X, Y, Z be the vector fields on $\mathbb{R}^3$ given by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

   (a) Show that the map $L\colon \mathbb{R}^3 \to \Gamma(\mathbb{R}^3)$ deined by $(a, b, c) \mapsto aX + bY + cZ$ is an isomorphism from $\mathbb{R}^3$ onto a subspace of the vector space $\Gamma(\mathbb{R}^3)$ of vector fields on $\mathbb{R}^3$.
   (b) Show that, under this isomorphism, the bracket of vector fields corresponds to the cross product of vectors in $\mathbb{R}^3$; that is, show $L((a, b, c) \times (d, e, f)) = [L(a, b, c), L(d, e, f)]$.
   (c) Compute the flow of the vector field $aX + bY + cZ$.