Math 3200: Intro to Topology – Homework 11

Due (at the start of class): Thursday, April 7

This assignment has 10 questions for a total of 100 points. Justify all your answers.

If (X, d) is a metric space and A, B are subsets, let us define the **distance between** A **and** B as

$$d(A, B) := \text{glb}\{d(a, b) \mid a \in A \text{ and } b \in B\}$$

where glb is the greatest lower bound. (Note that this definition makes sense because the set $\{d(a, b) \mid a \in A \text{ and } b \in B\} \subset \mathbb{R}$ is bounded below by 0.)

1. (10 pts) Let (X, d) be a metric space and A a subset of X. Prove that the function $f_A : X \to \mathbb{R}$ defined by $f_A(x) = d(\{x\}, A)$ is continuous.

2. (10 pts) Let K be a compact subset of a metric space (X, d). Prove that for any $p \in X$, there exist points $a, b \in K$ such that $d(p, a) \leq d(p, k)$ and $d(p, b) \geq d(p, k)$ for all $k \in K$. (That is, a and b are, respectively, *closest* and *furthest* points from p in K.)

3. (10 pts) Let (X, d) be a metric space. Prove that if A and B are disjoint compact subsets of X, then $d(A, B) > 0$.

4. (10 pts) Provide an example of a closed sets A and B in a metric space (X, d) such that A and B are disjoint but $d(A, B) = 0$.

5. (10 pts) Let X be an infinite set with the finite complement topology, and let A be an infinite subset of X. Describe A' and $\text{Cl}(A)$.

A topological space is called **limit point compact** if every infinite subset of X has a limit point.

6. (10 pts) Suppose that X is a limit point compact space and that A is a closed subset of X. Does it follow that A is limit point compact in the subspace topology? Provide a proof or counterexample.

7. (10 pts) If X is limit point compact and $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact? Provide a proof or counterexample.

8. (10 pts) Suppose that $f : X \to Y$ is a function between metric spaces (X, d_X) and (Y, d_Y) such that $d_Y(f(a), f(b)) \leq d_X(a, b)$ for all $a, b \in X$. (Such a function is said to be 1–**Lipschitz**.) Prove that f is continuous.

A set C in Euclidean space $\mathbb{R}^n$ is said to be **convex** if C contains every "line" between points of C, that is, if for all $x, y \in C$ and $t \in [0, 1]$ one has $(1 - t)x + ty \in C$.

9. (10 pts) Consider $\mathbb{R}^n$ with the standard Euclidean metric. Prove that for all $x \in \mathbb{R}^n$ and $\epsilon > 0$ the open ball $B(x, \epsilon)$ is convex.

A function $f : X \to X$ from a metric space (X, d) to itself is called a **contraction** if there is a number positive number $\lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

10. (10 pts) Suppose that (X, d) is a compact metric space and that $f : X \to X$ is a contraction. Prove that f has a unique fixed point. That is, prove there is a unique $p \in X$ such that $f(p) = p$. (*Hint*: Define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Consider the intersection of the sets $A_n = f^n(X)$.)