Math 3200: Intro to Topology – Homework 10
Due (at the start of class): Thursday, March 31

This assignment has 9 questions for a total of 100 points. Justify all your answers.

1. (10 pts) Let (X, d) be a metric space. Recall that a subset $A \subset X$ is said to be *bounded* if there exists $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in A$. Prove: A subset A is bounded in X if and only if there exist a point $x_0 \in X$ and a radius $R > 0$ such that $A \subset B(x_0, R)$.

2. (10 pts) We know that every bounded subset of $\mathbb{R}$ has a least upper bound and a greatest lower bound. Using this, prove that every bounded, increasing sequence in $\mathbb{R}$ converges. (Here, a sequence $\{x_n\}_{n=1}^{\infty}$ is *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and is *bounded* if there exists $K \in \mathbb{R}$ such that $x_n \leq K$ for all $n \in \mathbb{N}$.)

3. (10 pts) Prove or give a counter example: If A is a subset of a metric space X, then the derived set A' is closed.

Recall from Problem 2 on the Midterm Exam Part II, that the *arithmetic progression topology on the integers* $\mathbb{Z}$ is the topology generated by the collection $\mathcal{A}$ of all arithmetic progressions, and that an *arithmetic progression in* $\mathbb{Z}$ is a set of the form

$$P_{a,b} = \{a + kb \mid k \in \mathbb{Z}\} = \{\dots, a - 2b, a - b, a, a + b, a + 2b, \dots\}, \qquad \text{where } a, b \in \mathbb{Z} \text{ and } b \neq 0.$$

4. (10 pts) The Dirichlet Prime Number Theorem indicates that if a and b are relatively prime integers, then the arithmetic $P_{a,b}$ contains infinitely many prime numbers. Use this result to prove that $\mathbb{Z}$ in the arithmetic progression topology is not compact.

5. (10 pts) Let X be a compact topological space, and let $\{C_i\}_{i \in \mathbb{N}}$ be a collection of nonempty closed sets in X satisfying $C_{i+1} \subset C_i$ for each $i \in \mathbb{N}$. Prove that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

6. (10 pts) Let X and Y be topological spaces. Prove that if $X \times Y$ is compact, then so are X and Y.

7. Problem 4 from Homework 9 (last week) proves that if $f\colon X \to Y$ is a continuous bijective function where X is compact and Y is Hausdorff, then f is a homeomorphism.
   (a) (10 pts) Provide an example where $f\colon X \to Y$ is a continuous bijective function and X is compact, but f is not a homeomorphism.
   (b) (10 pts) Provide and example where $f\colon X \to Y$ is a continuous bijective function and Y is Hausdorff, but f is not a homeomorphism.

8. (10 pts) Does the "Nested Interval Lemma" still hold for nonempty nested closed subsets of $\mathbb{R}$? Specifically, prove or disprove the following: If $\{A_n\}_{n \in \mathbb{N}}$ is a collection of nonempty closed sets in $\mathbb{R}$ such that $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} A_n$ is nonempty.

9. (10 pts) Show that $[0, 1]$ is not a compact subset of $\mathbb{R}$ with the lower limit topology.