Math 3200: Intro to Topology – Homework 9

Due (at the start of class): Thursday, March 24

This assignment has 8 questions for a total of 90 points. Justify all your answers.

1. (10 pts) Prove: If A and B are compact subsets of a topological space X, then $A \cup B$ is compact.

2. (10 pts) Give an example of a collection of compact sets $A_1, A_2, A_3, \dots$ in $\mathbb{R}^2$ such that $\bigcup_{i=1}^{\infty} A_i$ is not compact.

3. (10 pts) Prove: If A and B are compact subsets of a Hausdorff topological space X, then $A \cap B$ is compact.

4. (10 pts) Suppose X and Y are compact, Hausdorff topological spaces and that $f \colon X \to Y$ is a continuous surjection. Prove that a subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open.

5. Consider the topological space $\mathbb{R}_{fc}$ (that is, the real line equipped with the finite complement topology) and let $A \subset \mathbb{R}_{fc}$ be any subset. Prove that A is compact.

6. (10 pts) Consider $\mathbb{R}^2$ with the standard topology (coming from the Euclidean metric). Let $p = (0, 0)$ be the origin and let $A = \overline{B}(p, 1) \setminus \{p\}$ be the closed unit ball minus the origin. Prove or disprove: A is compact.

7. (10 pts) Consider the topology on $\mathbb{Z}$ generated by the basis $\mathcal{B} = \{(-n, n) \mid n \in \mathbb{N}\}$.
   (a) (10 pts) Determine whether or not $(-5, 5) = \{-5, -4, \dots, 4, 5\}$ is a compact subsets of $\mathbb{Z}$ in this topology. (Justify your answer.)
   (b) (10 pts) Determine whether or not $\mathbb{Z}$ is compact in this topology. (Justify your answer.)

Recall that *Binary space* is the metric space (B, d), where B is the set of all infinite sequences of 0's and 1's (so elements of B have the form $x = (x_1, x_2, x_3, \dots)$ with $x_i \in \{0, 1\}$ for all $i \in \mathbb{N}$), and the distance between two points $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in B is defined to be

$$d(x, y) = \begin{cases} 0, & x = y \\ 1/n, & n = \min\{i \in \mathbb{N} \mid x_i \neq y_i\} \end{cases}$$

8. (10 pts) Show that Binary space B is totally disconnected, that is: Prove that if Y is a subset of B such that Y has more than one point, then Y is not connected.