**Math 3200: Intro to Topology – Homework 8**
Due (at the start of class): Thursday, March 17

**This assignment has 9 questions for a total of 90 points. Justify all your answers.**

*Several questions on this assignment use the fact that **all intervals in $\mathbb{R}$ are connected**

1. (10 pts) Prove: If $X$ is a topological space and for every $x, y \in X$ there exists a connected subset $C \subset X$ with $x, y \in C$, then $X$ is connected.

2. (10 pts) Let $X$ be a topological space. Prove: If $X = A \cup B$, $A \cap B \neq \emptyset$, and $A$ and $B$ are both connected, then $X$ is connected.

3. (10 pts) Let $X$ be an infinite set equipped with the finite complement topology. Prove or disprove: $X$ is connected.

A topological space is called **totally disconnected** if its only connected subsets are one-point sets.

4. (10 pts) Let $\mathbb{R}_\ell$ denote the set $\mathbb{R}$ equipped with the lower-limit topology. Prove or disprove: $\mathbb{R}_\ell$ is totally disconnected.

5. (10 pts) Show that no two of the spaces $(0, 1)$, $(0, 1]$, and $[0, 1]$ are homeomorphic. (*Hint*: What happens if you remove a point from each of these spaces?)

6. (10 pts) Let $X$ be a topological space with the following property: For every $a, b \in X$ there exits a continuous function $f \colon [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. Prove that $X$ is connected.

A topological space $X$ has the **fixed point property** if for every continuous function $f \colon X \to X$ there exist a point $x \in X$ such that $f(x) = x$. Such a point $x$ is called **fixed point** of $f$.

7. (10 pts) Prove that the fixed point property is a topological property. That is, if $X$ has the fixed point property and $h \colon X \to Y$ is a homeomorphism, then $Y$ has the fixed point property.

8. (10 pts) Show that the interval $[0, 1]$ has the fixed point property.

9. (10 pts) Does $[0, 1)$ have the fixed point property? Prove your answer.