**Math 3200: Intro to Topology – Homework 7**

Due (at the start of class): Thursday, March 3

**This assignment has 7 questions for a total of 100 points. Justify all your answers.**

1. Recall that Binary space $B$ is the metric space $(B, d)$, where $B$ is the set of all infinite sequences $(x_1, x_2, \dots)$ with each $x_i \in \{0, 1\}$, and the metric is defined by $d(x, y) = 0$ if $x = y$ and otherwise $d(x, y) = 1/\min\{i \mid x_i \neq y_i\}$. Let $Z \subset B$ be the subset consisting of all sequences that are eventually zero (see Homework 6 problem 4).
   (a) (10 pts) Prove or disprove: $Z$ is open.
   (b) (10 pts) Prove or disprove: $Z$ is closed.
   (c) (10 pts) What is $Z'$, the set of all limit points of $Z$? (Prove your answer, of course).

Recall that if $X$ is a topological space and $Y \subset X$, then a subset $A \subset Y$ is said to be **open in** $Y$ if $A$ is an open set in the subspace topology on $Y$. Similarly, $A$ is said to be **closed in** $Y$ if $A$ is closed in the subspace topology on $Y$.

2. Let $X$ be a topological space and let $Y \subset X$ be a subset. Give $Y$ the subspace topology.
   (a) (10 pts) Prove that $C \subset Y$ is closed in $Y$ if and only if $C = D \cap Y$ for some closed set $D$ in $X$.
   (b) (10 pts) Suppose $A$ is open in $Y$ and $Y$ is open in $X$. Prove that $A$ is open in $X$.

3. (10 pts) Let $X$ be a topological space and let $Y \subset X$ be a subset of $X$. Suppose $A \subset Y$ and $p \in Y$. Prove that $p$ is a limit point of $A$ with respect to the subspace topology on $Y$ if and only if $p$ is a limit point of $A$ with respect to the original topology on $X$.

4. (10 pts) Let $X$ and $Y$ be topological spaces. Suppose that $\mathcal{C}$ is a basis for $X$ and that $\mathcal{D}$ is a basis for $Y$. Prove that the collection

$$\mathcal{E} = \{C \times D \mid C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$$

   is a basis for a topology on $X \times Y$ *and* that basis generates the product topology on $X \times Y$.

5. (10 pts) Show that if $X$ and $Y$ are Hausdorff topological spaces, then so is the product $X \times Y$.

6. (10 pts) Let $X$ and $Y$ be topological spaces, and assume $A \subset X$ and $B \subset Y$. Prove that the subspace topology on $A \times B$ (considered as a subset of $X \times Y$) is the same as the product topology on $A \times B$ (where $A$ has the subspace topology inherited from $X$ and $B$ has the subspace topology inherited from $Y$).

7. (10 pts) Is the finite complement topology on $\mathbb{R}^2$ the same as the product topology on $\mathbb{R}^2$ that results from taking the product $\mathbb{R}_{fc} \times \mathbb{R}_{fc}$, where $\mathbb{R}_{fc}$ denotes the set $\mathbb{R}$ with the finite complement topology? Justify your answer.