Math 3200: Intro to Topology – Homework 6

Due (at the start of class): Thursday, February 25

This assignment has 10 questions for a total of 110 points. Justify all your answers.

1. (10 points) Suppose that $f\colon X \to Y$ is an injective function. Prove that $f^{-1}(f(A)) = A$ for every subset $A \subset X$.

2. (10 points) Suppose that that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions such that $g \circ f$ is bijective and g is injective. Prove that f is bijective.

3. (10 points) Prove that if $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $A \subset Z$ is a subset, then
$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A).$$

For **Questions 4–5** below, recall that Binary space is the set B of all infinite sequences of the form $x = (x_1, x_2, x_3, \dots)$, where $x_n \in \{0, 1\}$ for each $n \in \mathbb{N}$. In class we saw that B is uncountable.

4. (10 points) Let $Z \subset B$ be the subset of Binary space consisting of all sequences that are eventually 0. That is Z consists of all points $x = (x_1, x_2, x_3, \dots)$ in B with the property that there exists $K \in \mathbb{N}$ such that $x_i = 0$ for all $i \geq K$. Is Z countable or uncountable? Justify your answer.

5. (10 points) Recall that Fibonacci space F is the subset of B consisting of all points $x = (x_1, x_2, x_3, \dots)$ in B such that if $n \in \mathbb{N}$ and $x_n = 1$ then $x_{n+1} = 0$? Prove that F is uncountable by describing a bijection $h\colon B \to F$. (It would be tedious to write down a precise *formula* for h; instead just give a clear explanation of how the function h is defined an why it is bijective).

For the remaining questions, recall that a function $f\colon X \to Y$ between topological spaces is *continuous* if $f^{-1}(U)$ is open for every open set $U \subset Y$.

6. (10 points) Using the above definition, prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous (where $\mathbb{R}$ is given the standard topology).

7. (10 points) Suppose $f\colon X \to Y$ is a continuous function, $A \subset X$, and $p \in X$. Prove that if p is a limit point of A and $f(p) \notin f(A)$, then $f(p)$ is a limit point of $f(A)$.

Recall that a function $f\colon X \to Y$ is said to be *continuous at the point* $p \in X$ if for every neighborhood U of $f(p)$ there exists a neighborhood V of p such that $f(V) \subset U$.

8. (10 points) Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point (where $\mathbb{R}$ has the standard topology).

9. (10 points) Let $f\colon X \to Y$ be a function of topological spaces. Prove that f is continuous if and only if f is continuous at p for each point $p \in X$.

10. Let X be a topological space, and suppose that $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are both continuous functions (where $\mathbb{R}$ has the standard topology).
    (a) (10 points) Prove that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
    (b) (10 points) Let $h\colon X \to \mathbb{R}$ be the function $h(x) = \min\{f(x), g(x)\}$. Prove h is continuous.