Math 3200: Intro to Topology – Homework 5

Due (at the start of class): Thursday, February 11

This assignment has 8 questions for a total of 100 points. Justify all your answers.

1. (10 points) Prove or give a counter example: In a metric space (X, d), for each $\epsilon > 0$ and $p \in X$, the closed ball $\overline{B}(p, \epsilon)$ is the closure of the open ball $B(p, \epsilon)$.

2. (10 points) Let (X, d) be a metric space and let $A \subset X$ be a subset. Prove that A is closed in X if and only if whenever $\{p_i\}_{i=1}^{\infty}$ is a sequence in A that converges to a point $q \in X$, it follows that $q \in A$.

For **Questions 3–4**, recall that a topological space $(X, \mathcal{T})$ is said to be **Hausdorff** if for all $p, q \in X$ with $p \neq q$, there exists a neighborhood U of p and a neighborhood V of q such that $U \cap V = \emptyset$.

3. (10 points) Let X be a Hausdorff topological space. Prove that if $\{p_i\}_{i=1}^{\infty}$ is a sequence in X that converges to the point $p \in X$ and also converges to the point $q \in X$, then $p = q$.

4. (10 points) Prove or give a counterexample: If A is a subset of a Hausdorff topological space X and p is a limit point of $\overline{A}$, then p is a limit point of A.

For **Question 5**, Recall that a set S of real numbers is said to be **bounded below** if there exists $b \in \mathbb{R}$ such that $b \leq x$ for all $x \in S$. Any such number b is said to be a **lower bound** for S. A lower bound $g \in \mathbb{R}$ is said to be a **greatest lower bound** for S if $g \geq b$ for every lower bound b of S. It is a property of the real numbers that every subset $S \subset \mathbb{R}$ that is bounded below has a unique greatest lower bound. This greatest lower bound is called the **infimum** of S and denoted $\inf(S)$. Note that $\inf(S)$ is characterized by the property that $\inf(S)$ is a lower bound and that for all $\epsilon > 0$, $\inf(S) + \epsilon$ is NOT a lower bound for S.

5. (10 points) Let A be a subset of a metric space (X, d). For $p \in X$, define

$$d(p, A) := \inf\{d(p, x) \mid x \in A\}.$$

(Note that the infimum exists because 0 is a lower bound for $\{d(p, x) \mid x \in A\}$). Prove that $d(y, A) = 0$ if and only if $y \in \overline{A}$.

6. (10 points) Prove or give a counter example: If C is a non-empty, closed subset of a metric space (X, d) and $p \in X \setminus C$, then there exists $q \in C$ such that $d(p, q) \leq d(p, x)$ for all $x \in C$.

7. Choose a relation $\subset$, $\supset$, or $=$ to place in each blank below so that the statement holds for every function $f : X \to Y$ between sets X and Y. Justify your answer. Moreover, if your answer is not "=", then provide an example showing that equality need not hold.
   (a) (10 points) For every subset $A \subset X$, one has A ___ $f^{-1}(f(A))$.
   (b) (10 points) For every subset $B \subset Y$, one has B ___ $f(f^{-1}(B))$.
   (c) (10 points) For all subsets $A_1, A_2 \subset X$, one has $f(A_1) \cap f(A_2)$ ___ $f(A_1 \cap A_2)$.

8. (10 points) Let X bet a three-element, set $X = \{a, b, c\}$. Equip X with the topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $U = \{a, b\}$ and $V = \{c\}$. What are U', $\overline{U}$, V', and $\overline{V}$?