Math 3200: Intro to Topology – Homework 4

Due (at the start of class): Thursday, February 4

This assignment has 4 questions for a total of 110 points. Justify all of your answers.

1. In $\mathbb{R}$, let $\mathcal{B}_\ell$ be the collection

$$\mathcal{B}_\ell := \{[a,b) \subset \mathbb{R} \mid a, b \in \mathbb{R} \text{ and } a < b\}.$$

   (a) (10 points) Prove that $\mathcal{B}_\ell$ is a basis for a topology on $\mathbb{R}$.

   (b) (10 points) Let $\mathbb{R}_\ell$ denote the set $\mathbb{R}$ equipped with the topology generated by $\mathcal{B}_\ell$; we call this the *lower limit topology.* Determine whether the lower limit topology is *finer, coarser, equal to, or not comparable to* the standard topology on $\mathbb{R}$.

   (c) (10 points) In the topological space $\mathbb{R}_\ell$, determine the set of limit points of $A = (0,1) \cup \{2\}$.

2. (10 points) Recall that a subset A in a topological space X is said to be **dense** if $\bar{A} = X$. Prove that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$ (with the standard topology).

3. Choose a relation $\subset$, $\supset$, or $=$ to place in each blank below so that the statement holds for every topological space X and all subsets $A, B \subset X$. Justify your answer. Moreover, if your answer is not "=", then provide an example showing that equality need not hold.

   (a) (10 points) $\mathrm{Cl}(X \setminus A)$ ___ $X \setminus \mathrm{Int}(A)$

   (b) (10 points) $\overline{A} \cup \overline{B}$ ___ $\overline{A \cup B}$

   (c) (10 points) $\overline{A} \cap \overline{B}$ ___ $\overline{A \cap B}$

4. (40 points) Let A and B be subsets of a topological space X. For each statement below, either provide a proof (if it is true) or a counterexample (if it is false).

   (a) If p is a limit point of A or p is a limit point of B, then p is a limit point of $A \cup B$.

   (b) If p is a limit point of $A \cap B$, the p is a limit point of A and p is a limit point of B.

   (c) If p is a limit point of A and p is a limit point of B, then p is a limit point of $A \cap B$.

   (d) If p is a limit point of $A \cup B$, then p is a limit point of A or p is a limit point of B.