Math 3200: Intro to Topology – Homework 2
Due (at the start of class): Thursday, January 21

This assignment has 5 questions for a total of 60 points.

1. (10 points) Fix real numbers $a < b$, and let $\mathcal{F}$ be the set of all continuous functions $f\colon [a,b] \to \mathbb{R}$. For $f, g \in \mathcal{F}$, define
$$d(f,g) := \int_a^b |f(x) - g(x)|\, dx.$$
Using appropriate theorems from calculus, prove that $(\mathcal{F}, d)$ is a metric space.

In the problems below, let d denote the standard Euclidean distance function on $\mathbb{R}^n$.

2. (10 points) If $p, q \in \mathbb{R}^n$, $p \neq q$, and $r = \frac{1}{2}d(p,q)$, prove that
$$\overline{B}(p,r) \cap B(q,r) = \emptyset.$$

3. (10 points) Suppose $p, q \in \mathbb{R}^n$ are such that $p \neq q$. Prove that there exits $r > 0$ such that
$$\overline{B}(p,r) \cap \overline{B}(q,r) = \emptyset$$

4. (10 points) For $x, y \in \mathbb{R}^n$ and $0 \leq t \leq 1$, set $p := x + t(y - x)$. Prove that
$$d(x,p) + d(p,y) = d(x,y).$$

5. Let d be the standard Euclidean distance on $\mathbb{R}^2$ (as above), and let d_T be the *taxicab* distance function on $\mathbb{R}^2$ defined by
$$d_T(x,y) = |x_1 - y_1| + |x_2 - y_2|\,, \quad \text{for } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$
   (a) (10 points) Prove that for all $x, y \in \mathbb{R}^2$ one has:
$$\frac{1}{2}\, d_T(x,y) \leq d(x,y) \leq \sqrt{2}\, d_T(x,y).$$
   (*Hint:* Think about the maximum metric d_M.)

   (b) (10 points) Let A be a subset of $\mathbb{R}^2$. Prove that A is open in the metric space $(\mathbb{R}^2, d)$ if and only if A is open in the metric space $(\mathbb{R}^2, d_T)$.