Math 3200: Intro to Topology – In-class presentation challenge problems

Here* are some challenge problems. Please let me know if you would like to volunteer to present a **clear and well-thought-out** solution to the class. Good presentations will earn bonus points for the homework.

I – General questions

1. Let X be an uncountable set with the countable complement topology and let $A \subset X$. Describe A' and $\text{Int}(A)$ and $\text{Cl}(A)$ in the cases that (i) A is finite, (ii) A is countably infinite, and (iii) A is uncountable.
2. Prove or give a counterexample: If A is a subset of a topological space X, then the derived set A' is closed.
3. Does Sequence Proposition 2 ($q \in \overline{A} \iff$ there is a sequence in A converging to q) hold in arbitrary topological spaces? Provide a proof or counterexample.
4. Does Sequence Proposition 3 ($q \in A' \iff$ there is a sequence with distinct terms in A converging to A) hold in arbitrary Hausdorff spaces? Provide a proof or counterexample.
5. Is Fibonacci space homeomorphic to Binary space?
6. Let B be Binary space. Is B homeomorphic to $B \times B$? **Note:** This problem was mostly explained in class, though some details were omitted. If you would still like to present this problem (for perhaps less bonus than other problems), please give a very clear and to-the-point presentation that fills in all the details.
7. Prove or give a counterexample: If A and B are compact subsets of a topological space, then $A \cap B$ is compact.
8. Prove or give a counterexample: If C is a non-empty, closed subset of $\mathbb{R}^n$ (with the standard metric) and $p \in \mathbb{R}^n \setminus C$, then there exists $q \in C$ such that for every $x \in C$, $d(p, q) \leq d(p, x)$. (That is, there is a point of c closest to p.)
9. Prove or disprove. If X is a topological space, then following statements are equivalent:
    1. For every $p \in X$, $\{p\}$ is closed in X.
    2. For every $A \subset X$, the derived set A' is closed in X.
10. If X is a compact and Hausdorff topological space, is it true that every sequence in X has a convergent subsequence? Provide a proof or counterexample.

*This list is from April 5, 2016; for the most up-to-date and expansive list, check the course webpage

II – Variation on countable complement topology

Let X be an uncountable set and fix a point $p \in X$. Define $\mathcal{T}$ to be the set of all subsets U of X such that either

- $p \in U$ and $X \setminus U$ is countable, or
- $p \notin U$

Prove or disprove the following:

11. $\mathcal{T}$ is a topology.

12. $\mathcal{T}$ is closed under countable intersections

13. $(X, \mathcal{T})$ is Hausdorff

14. No sequence of distinct terms in X (with the topology $\mathcal{T}$) converges.

15. The point p is a limit point of X (with the topology $\mathcal{T}$). In fact $X' = \{p\}$

III – Answer Space

Let B be Binary space and let $S = \{y, n\}$ be a set with exactly two elements ("yes" and "no"). Define Answer space A to be the set of all functions from B to S. That is $A = \{f \colon B \to \{y, n\}\}$.

(Intuitively, any question q can be encoded into an infinite string of 0's and 1's; that is q is a point of B. If $f \in A$, then $f(q) \in S$ is an answer to the question q. Thus A consists of all answers to all "yes"–or–"no" questions.)

Define a topology on A as follows. First, a *basic open subset of* A is a set of the form $U_{(F,f)}$, where F is a finite subset of B, $f \in A$, and $g \in A$ is in $U_{(F,f)}$ if and only if $g(x) = f(x)$ for all $x \in F$. The, open sets in A are those that are arbitrary unions of basic open sets.

Define the *Yes Man* to be the element $\gamma \in A$ such that $\gamma(x) = y$ for all $x \in B$. Define the *Mostly No* subspace M of A by $\varphi \in M$ if and only if $\varphi(x) = n$ for all but finitely many $x \in B$.

Prove or disprove the following:

16. $U_{(F,f)} = U_{(F,g)}$ for all $g \in U_{(F,f)}$.

17. $U_{(F_1,f)} \cap U_{(F_2,f)} = U_{(F_1 \cap F_2, f)}$

18. If we include the empty set in the topology defined above, then we have indeed defined a topology on A. (In the remaining problems, we understand A to be endowed with the topology just described).

19. A is Hausdorff

20. $\gamma \in \overline{M}$

21. No sequence in M converges to γ

22. Extra extra credit: A is compact

IV – Answer Space with a different topology

Suppose we define a different topology on A by replacing the finite subsets F of B in the discussion above by *compact* subsets F of B.

23. Investigate all of the questions in the previous problem with respect to this new topology on A (if indeed it is a topology).

V – Order completeness of Binary space

Define a relation $\leq$ on Binary space B as follows. If $x = (x_1, x_2, x_3. \dots)$ and $y = (y_1, y_2, y_3, \dots)$ are points of B, then $x \leq y$ if and only if

- $x = y$, or
- $x \neq y$ and $x_n < y_n$ where $n = \min\{i \in \mathbb{N} \mid x_i \neq y_i\}$

24. Verify that $\leq$ satisfies the following properties:

    1. For every $x, y \in B$, either $x \leq y$ or $y \leq x$.
    2. For every $x, y \in B$, if $x \leq y$ and $y \leq x$, then $x = y$.
    3. For every $x, y, z \in B$, if $x \leq y \leq z$, then $x \leq z$.

25. Show that $(B, \leq)$ is *order complete* in the following sense. If $\{x_i\}_{i=1}^{\infty}$ is a sequence of points in B that is increasing (i.e., $x_1 \leq x_2 \leq x_3 \leq \cdots$), then $\{x_i\}_{i=1}^{\infty}$ converges in B.

26. Using the binary representation of real numbers and the order completeness of B, prove that $[0, 1]$ is order complete.