Math 7210: Riemannian Geometry – Homework 2

Due in class: Wednesday, September 4, 2019

1. (from #0.5 of do Carmo) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$f(x,y,z) = (x^2 - y^2, xy, xz, yz), \qquad (x,y,z) = p \in \mathbb{R}^3.$$

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with the origin $0 \in \mathbb{R}^3$. Observe that the restriction $\varphi = F|_{S^2}$ is such that $\varphi(p) = \varphi(-p)$, and consier the mapping $\tilde{\varphi}\colon P^2(\mathbb{R}) \to \mathbb{R}^4$ given by

$$\tilde{\varphi}([p]) = \varphi(p), \quad [p] = \text{ equiv. class of } p = \{p, -p\}.$$

Prove that

(a) $\tilde{\varphi}$ is an immersion.

(b) $\tilde{\varphi}$ is injective; together with (a) and the compactness of $P^2(\mathbb{R})$, this implies that $\tilde{\varphi}$ is an embedding.

2. (from #0.8 of do Carmo) Let M and N be differentiable manifolds and let $\varphi\colon M \to N$ be a local diffeomorphism. Prove that if N is orientable, then M is orientable.

3. (from #0.9 of do Carmo) Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M.

(a) Prove that the manifold M/G (Example 4.8) is orientable if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

(b) Use (a) to show that the projective plane $P^2(\mathbb{R})$, the Klein bottle and the Möbius band are non-orientable.

(c) Prove that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

4. Let X, Y, Z be the vector fields on $\mathbb{R}^3$ given by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(a) Show that the map $L\colon \mathbb{R}^3 \to \Gamma(\mathbb{R}^3)$ deined by $(a,b,c) \mapsto aX + bY + cZ$ is an isomorphism from $\mathbb{R}^3$ onto a subspace of the vector space $\Gamma(\mathbb{R}^3)$ of vector fields on $\mathbb{R}^3$.

(b) Show that, under this isomorphism, the bracket of vector fields corresponds to the cross product of vectors in $\mathbb{R}^3$; that is, show $L((a,b,c) \times (d,e,f)) = [L(a,b,c), L(d,e,f)]$.

(c) Compute the flow of the vector field $aX + bY + cZ$.