Math 7210: Riemannian Geometry – Homework 1
Due in class: Wednesday, August 28, 2019

For M a smooth manifold and $p \in M$, recall that $C_p^\infty(M)$ denotes the set of *germs of smooth functions at* p. That is, $C_p^\infty(M)$ is the set of equivalence classes of pairs (U, f), where $U \subset M$ is an open neighborhood of p and $f \colon U \to \mathbb{R}$ is smooth, and two pairs $(U, f), (V, g)$ are equivalent if f and g agree on a neighborhood of p. Recall that a *derivation of* M *at* p is a linear function $D \colon C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = f(p)D(g) + g(p)D(f), \qquad \text{for all } f, g \in C_p^\infty(M).$$

The set of derivations of M at p is an $\mathbb{R}$–vector space and is denoted $\mathcal{D}_p^\infty(M)$.

1. Let M be a smooth manifold and let $p \in M$ be a point. Show that if $D \colon C_p^\infty(M) \to \mathbb{R}$ is a derivation of M at p, then $D(c) = 0$ for any constant function c.

From calculus, we know that if $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable at a, then we may write $f(x) = f(a) + g(x)(x - a)$ for some function $g \colon \mathbb{R} \to \mathbb{R}$ with $g(a) = f'(a)$. In higher dimensions, Taylor's theorem with remainder says that for any smooth function $f \colon B_r(p) \to \mathbb{R}$ defined on the ball of radius $r > 0$ about $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, we may write

$$f(x) = f(p) + \sum_{i=1}^{n} g_i(x)(x_i - p_i)$$

for some smooth functions $g_1, \ldots, g_n \colon B_r(p) \to \mathbb{R}$ whose values at p are given by $g_i(p) = \frac{\partial f}{\partial x_i}(p)$.

2. Let $p = (p_1, \ldots, p_n)$ be a point in the manifold $\mathbb{R}^n$. We saw in class that for each $i = 1, \ldots, n$, the i^{th} partial derivative $\frac{\partial}{\partial x_i}$ (at p) is a derivation of $\mathbb{R}^n$ at p. In this problem you will prove that the set $\beta = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ is a basis for the vector space $\mathcal{D}_p(\mathbb{R}^n)$ of derivations at p.
   (a) (easy) Show that β is a linearly independent subset of $\mathcal{D}_p(M)$.
   (b) (harder) Show that β spans $\mathcal{D}_p(M)$.
   (*Hint:* Use Taylor's theorem with remainder, above.)

3. (Exercise 0.1 of do Carmo) Let M and N be differentiable manifolds, and let $\{(U_\alpha, \mathbf{x}_\alpha)\}$, $\{(V_\beta, \mathbf{y}_\beta)\}$ be differentiable structures on M and N, respectively. Consider the cartesian product $M \times N$ and the mappings $\mathbf{z}_{\alpha\beta}(p, q) = (\mathbf{x}_\alpha(p), \mathbf{y}_\beta(q))$, $p \in U_\alpha$, $q \in V_\beta$. Prove that $\{(U_\alpha \times V_\alpha, \mathbf{z}_{\alpha\beta})\}$ is a differentiable structure on $M \times N$ in which the projections $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ are differentiable. With this differentiable structure $M \times N$ is called the *product manifold* of M with N.