MATH 6100-A, FALL 2017
THEORY OF FUNCTIONS OF REAL VARIABLES
MWF 9:10–10:00, SC1120

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Office Hours, MWF 10:00-11:00 or by Appointment

Course Description

This is an introduction to Lebesgue integration and related topics (see syllabus below). It assumes a working knowledge of Advanced Multivariate Calculus, various notions of convergence of sequences and series and a working knowledge of basic notions of Topology and Number Theory (see below).

TEXTBOOK AND SUGGESTED BOOKS

No textbook will be required. I will pass along class notes which should be sufficient. The list of books below is meant to complement the lectures and for consultation. They all have been placed in **reserve** in the Sc. Library. While I will follow mainly [1], material will be taken out of the others.

[1]. E. DiBenedetto, *Real Analysis, 2nd Ed.*, Birkhäuser, Boston, 2016, ISBN-978-1-4939-4003-5

[2]. H.L. Royden, *Real Analysis*, McMillan, ISBN-0-02-404151-3

[3]. A. Friedman, *Foundations of Modern Analysis*, Holt Reinhart& Winston, ISBN 03-081291-7

[4]. A. Zygmund and R.L. Weeden, *Measure and Integral*, Marcel Dekker, ISBN 0-8247-6499-4

[5]. W. Rudin, *Real and Complex Analysis*, McGraw Hill, ISBN 0-07-054234-1

[6]. P. Halmos, *Measure Theory*, Springer Verlag NY, ISBN 0-387-90088-8, ISBN 0-540-90088-8

[7]. C.L. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press 1992, ISBN 0-8493-7157-0

[8]. G.B. Folland, *Real Analysis, Modern Techniques and Their Applications*, Wiley-Interscience, 1984, ISBN 0-471-80958-6

[9]. E.H. Lieb and M. Loss, *Analysis*, Amer. Math. Soc. Vol. 14, 1996, ISBN 0-8218-0632-7

[10]. Soo Bong Chae, *Lebesgue Integration*, Springer Verlag, 1994, ISBN 0-387-94357-9

[11]. F. Riesz and B.S. Nagy, *Functional Analysis*, Dover 1990, ISBN 0-486-66289-6

[12]. S. Saks, *Theory of the Integral*, Dover, Phoenix Ed. 2005, ISBN 0-486-44648-4.

SYLLABUS FOR THE FALL SEMESTER 2017

Measuring Sets

1. Partitioning Open Subsets of $\mathbb{R}^N$
2. Limits of Sets, Characteristic Functions and σ-Algebras
3. Measures
4. Outer Measures and Sequential Coverings

   The Lebesgue Outer Measure in $\mathbb{R}^N$
   The Lebesgue-Stieltjes Outer Measure
5. The Hausdorff Outer Measure in $\mathbb{R}^N$
6. Constructing Measures from Outer Measures
7. The Lebesgue-Stieltjes Measure on $\mathbb{R}$
8. Borel Measures
9. The Hausdorff Measure in $\mathbb{R}^N$
10. Extending Measures from Semi-Algebras to σ-Algebras
11. Lebesgue-Stieltjes and Hausdorff Measures
12. Necessary and Sufficient Conditions of Measurability
13. More on Extensions from Semi-Algebras to σ-Algebras
14. The Lebesgue Measure of Sets in $\mathbb{R}^N$
15. A Necessary and Sufficient Condition of Measurability
16. A Non-Measurable Set
17. Borel Sets, Measurable Sets and Incomplete Measures

    A Continuous Increasing Function $f : [0, 1] \to [0, 1]$
    On the Pre-Image of a Measurable Set
18. More on Borel Measures

    Some Extensions to General Borel Measures
    Regular Borel Measures and Radon Measures
19. Regular Outer Measures and Radon Measures
20. Vitali Coverings
21. The Besicovitch Covering Theorem
22. **Complements:** Whitney Decomposition; Completion of a Measure space; Probability measures; The Hausdorff dimension of a set $E \subset \mathbb{R}^N$; Metric Outer Measures; Inner Measure and Measurability; Independent sets; the Borel-Cantelli Lemma; The Peano-Jordan Measure of Bounded Sets in $\mathbb{R}^N$; Simpler Forms of the Besicovitch Covering Theorem

The Lebesgue Integral

1. Measurable Functions
2. The Egorov Theorem
3. Approximating Measurable Functions by Simple Functions
4. Convergence in Measure
5. Quasi Continuous Functions and Lusin's Theorem
6. Integral of Simple Functions
7. The Lebesgue Integral of Nonnegative Functions
8. Fatou's Lemma and the Monotone Convergence Theorem
9. Basic Properties of the Lebesgue Integral
10. Convergence Theorems
11. Absolute Continuity of the Integral
12. Product of Measures
13. On the Structure of $(\mathcal{A} \times \mathcal{B})$
14. The Theorem of Fubini–Tonelli
15. The Tonelli Version of the Fubini Theorem
16. Some Applications of the Fubini-Tonelli Theorem

    Integrals in Terms of Distribution Functions
    Convolution Integrals
    The Marcinkiewicz Integral

17. Signed Measures and the Hahn Decomposition
18. The Radon-Nykodým Theorem
19. Decomposing Measures

    The Jordan Decomposition
    The Lebesgue Decomposition
    A General Version of the Radon-Nykodým Theorem

20. **Complements:** Comparing the Lebesgue integral with the Peano-Jordan integral; Other Versions of Dominated Convergence; Random variables; integrals of a random variable; some distributions; joint distributions; distribution functions; distribution function of a random variable; independent random variable; sequences of random variables; the Kolmogorov consistency theorem; Integrals in Terms of Distribution Functions

Topics on Measurable Functions of Real Variables

1. Functions of Bounded Variations
2. Dini Derivatives
3. Differentiating Functions of Bounded Variation
4. Differentiating Series of Monotone Functions
5. Absolutely Continuous Functions
6. Density of a Measurable Set
7. Derivatives of Integrals
8. Differentiating Radon Measures
9. Existence and Measurability of $D_\mu \nu$
10. Representing $D_\mu \nu$

    Representing $D_\mu \nu$ for $\nu \ll \mu$

    Representing $D_\mu \nu$ for $\nu \perp \mu$
11. The Lebesgue Differentiation Theorem

    Points of Density

    Lebesgue Points of an Integrable Function
12. Regular Families
13. Convex Functions
14. The Jensen's Inequality
15. Extending Continuous Functions
16. The Weierstrass Approximation Theorem
17. The Stone-Weierstrass Theorem
18. Proof of the Stone–Weierstrass Theorem
19. The Ascoli-Arzelà Theorem
20. **Complements:** A Continuous Nowhere Differentiable Function; The Cantor Ternary Function; A Strictly Monotone Function with a.e. Zero Derivative; The Function of the Jumps; Convex Functions in $\mathbb{R}^N$; The Legendre Transform; Finiteness and Coercivity; Discrete Versions of Jensen's Inequality; Precompact Subsets of $C(\bar{E})$; Applications of the Baire Category Theorem; General Versions of the Ascoli–Arzelà Theorem.

The $L^p(E)$ Spaces

1. Functions in $L^p(E)$ and their Norm
2. The Hölder and Minkowski Inequalities
3. The Reverse Hölder and Minkowski Inequalities
4. More on the Spaces $L^p(E)$ and their Norm

   Characterizing the Norm $\|f\|_{p,E}$ for $1 \le p < \infty$
   The Norm $\|\cdot\|_{\infty,E}$ for E of Finite Measure
   The Continuous Version of the Minkowski Inequality

5. $L^p(E)$ for $1 \le p \le \infty$ as Normed Spaces and Equivalence Classes
6. $L^p(E)$ for $1 \le p \le \infty$ as a Metric Topological Vector Space
7. Convergence in $L^p(E)$ and Completeness
8. Separating $L^p(E)$ by Simple Functions
9. Weak Convergence in $L^p(E)$
10. Weak Lower Semi-Continuity of the Norm in $L^p(E)$
11. Weak Convergence and Norm Convergence
12. Linear Functional in $L^p(E)$
13. The Riesz Representation Theorem
14. The Hanner and Clarkson Inequalities
15. Uniform Convexity of $L^p(E)$ for $1 < p < \infty$
16. The Riesz Representation Theorem by Uniform Convexity
17. If $E \subset \mathbb{R}^N$ and $p \in [1, \infty)$, then $L^p(E)$ is Separable
18. $L^\infty(E)$ is not Separable
19. Selecting Convergent Subsequences
20. Continuity of the Translation in $L^p(E)$
21. Approximating Functions in $L^p(E)$ with functions in $C^\infty(E)$
22. Characterizing Pre-Compact Sets in $L^p(E)$
23. **Complements:** The Spaces L^p for $0 < p < 1$; the Spaces L^q for $q < 0$; the spaces ℓ_p; Variants of the Hölder and Minkowski Inequalities; Remarks on Weak Convergence; Comparing the Various Notions of Convergence; A Metric Topology for $L^p(E)$ for $0 < p < 1$; open Convex Subsets of $L^p(E)$ when $0 < p < 1$; Bounded Linear Functional in $L^p(E)$ for $0 < p < 1$; Caloric and Harmonic Extensions; sequences of random variables; convergence of series of independent random variables; the Kolmogorov inequality; Kolmogorov strong law of large numbers; strong law of large numbers for equi-distributed random variables.

MORE ON PRE-REQUISITES

Exposure to the following topics will be needed:

Countability and Cardinality

Countable Sets, the Cantor Set; Cardinality; Cardinality of Some Infinite Cartesian Products; Orderings; Maximal Principle and the Axiom of Choice; Well Ordering; The First Uncountable; Generalized Cantor Set; Diadic Expansions; Perfect Sets.

Topologies and Metric Spaces

Topological Spaces; Hausdorff and Normal Spaces; Urysohn's Lemma; The Tietze Extension Theorem; Bases of a Topology; Axioms of Countability and Product Topologies; Compact Topological Spaces; Sequential Compactness; Compact Subsets of $\mathbb{R}^N$; Continuous Functions on Countably Compact Spaces; Product of Compact Spaces; Vector Spaces; Convex Sets; Linear Maps and Isomorphisms; Topological Vector Spaces; Boundedness and Continuity; Linear Functionals; Finite Dimensional Topological Vector Spaces; Locally Compact Spaces; Metric Spaces; Separation and Axioms of Countability; Equivalent Metrics; Pseudo Metrics; Metric Vector Spaces; Spaces of Continuous Functions; On the Structure of a Complete Metric Space; The Baire Category theorem; Compact and Totally Bounded Metric Spaces; The Box Topology; The Alexandrov One-point compactification; Hamel Bases; Dimension of a Vector Space; Locally Compact Spaces; Countable Products of Metric Spaces; Completion of a Metric Space; Versions of the Baire Category Theorem.

Note

A course in point-set Topology runs parallel to Real Analysis. The topics above do not have to be known at the outset but their knowledge will have to be acquired progressively and in parallel with Real Analysis.

GRADING POLICY

Throught the semester homework problems will be suggested, either out of [1] or other texts. There will be 3 midterm in class exams on Sept 22, Oct 20, Nov 17. Each midterm will be based on the material and suggested homework problems of the preceeding 4 weeks or so. Your exams will be graded in 100's. The average of your scores will be your final score. Your final grade will assigned as:

A 90–100

B 80–89.9

C 70–79.9

D 60–69.9

F 0–59.9