Lower Bounds for Cubature on the Sphere in Sobolev Spaces

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In this talk I will present results from recent joint work with Ian H. Sloan on cubature on the unit sphere $S^2$ in Sobolev spaces. We prove that the worst-case error $e(H^s; Q_m)$ of any $m$-point cubature rule $Q_m$ in the Sobolev space $H^s = H^s(S^2)$, $s > 1$, satisfies $e(H^s; Q_m) \geq c_s m^{-s/2}$, where the constant $c_s$ does not depend on $Q_m$ and on $m \in \mathbb{N}$. This estimate is sharp (optimal) because we can identify sequences $(Q_{m(n)})_{n \in \mathbb{N}}$ of $m(n)$-point cubature rules, with $\lim_{n \to \infty} m(n) = \infty$, that achieve this order of convergence, that is for which $e(H^s; Q_{m(n)}) \leq \tilde{c}_s m(n)^{-s/2}$ for all $n \in \mathbb{N}$, with $\tilde{c}_s$ independent of $n$.

This talk focuses on the proof of the lower bound $e(H^s; Q_m) \geq c_s m^{-s/2}$ for any $m$-point cubature rule. The idea is to construct a ‘bad’ function $f_m \in H^s$, such that the cubature error in $H^s$ for $f_m/\|f_m\|_{H^s}$ is bounded from below by $c_s m^{-s/2}$. This function is constructed geometrically in such a way that it vanishes in all points of the cubature rule, and the construction of $f_m$ uses results about sphere packing. A crucial step consists in delicately estimating the norm $\|f_m\|_{H^s}$.