Distribution Measures for Pointsets on the Sphere
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Let $S^d$ be the unit sphere in $\mathbb{R}^{d+1}$ and $x_1, \ldots, x_N$ be $N$ points in $S^d$ (for some notions used later, we will need that the points are pairwise distinct).

There are essentially two types of measures for quality of the distribution of $N$ points in $S^d$:

- “combinatorial” measures:
  - discrepancy:
    \[
    D_N^C(x_n) = \sup_{A \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_A(x_n) - \sigma_d(A) \right|
    \]
    where $\mathcal{C}$ is any “suitable” system of subsets of $S^d$, for instance caps or slices. $\sigma_d$ denotes the normalized surface measure.
  - dispersion:
    \[
    \delta_N(x_n) = \sup_{x \in S^d} \min_{k} |x - x_k|,
    \]
    i.e. the radius of the largest spherical cap not containing any point $x_n$

- “analytical” measures:
  - error in equal weight numerical integration:
    \[
    I_N(f) = \left| \sum_{n=1}^{N} f(x_n) - \int_{S^d} f(x) \, d\sigma_d(x) \right|
    \]
  - Lipschitz-discrepancy:
    \[
    \sup_{f \in \text{Lip}_1} I_N(f),
    \]
    where $\text{Lip}_1$ denotes the set of all continuous functions with $|f(x) - f(y)| \leq |x - y|$.
  - energy:
    \[
    \sum_{\substack{i,j=1 \\ i \neq j}}^{N} g(|x_i - x_j|),
    \]
    where $g$ is a monotonically decreasing function with a singularity at 0, for example $g(r) = r^{-\alpha}$.

We will discuss relations between these different measures and point out why it is especially difficult to relate “combinatorial” to “analytical” measures.