On a Conjecture concerning the Petersen Graph

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(joint work with Michael Plummer)
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Conjecture (Robertson, private communication)

If a graph $G$ is

(i) 3-connected and internally-4-connected;
(ii) girth is 5;
(iii) every cycle of length greater than 5 has a chord.

Then $G$ is the Peterson graph.

- (ii) and (iii) $\Rightarrow$ the only odd hole is of size 5

- (motivation?) raised this question when working on the
  (Strong) Perfect Graph Conjecture
  (perfect $\Leftrightarrow$ no odd hole and odd anti-hole)
Theorem (Nelson, Plummer, Robertson, Z, 2011)

(i) true if $G$ is cubic;

(ii) true if there is a 5-cycle $C$ whose neighbor $N(C)$ contains a 2-path;

(iii) For any 5-cycle $C$, $N(C)$ cannot be independent;

Therefore, for every 5-cycle $C$, the neighbor $N(C)$ consists of matching edges and isolated vertices.

Based on this structure, we come up with a counter-example to the conjecture.
Theorem (Plummer, Z)

There is a counter-example to Robertson’s conjecture on the Petersen graph.
Construction of the counterexample

(I) Heawood graph $H$

3-connected, internally-4-connected, bipartite, girth $= 6$
Construction of the counterexample

(II) Three copies of Heawood graph $3H$

(again, 3-connected, internally-4-connected, bipartite, girth = 6)
(III) A frame graph $G_0$

$G_0 = \text{Petersen} - \text{two matching edges (of distance 2)}$

($G_0 \cup \{ad, cf\} = \text{Petersen}$)
Construction of the counterexample

(IV) Adding 21 vertices to $G_0$
(V) Adding matching edges between some of these 21 vertices
Construction of the counterexample

(VI) Connecting four copies of $3H$ to $G_0$ through these 21 vertices

(to preserve \textit{bipartite, girth condition} and \textit{internally-4-connected})
The resulting graph $G$ is

- **girth 5**

- **3-connected and internally-4-connected**

- **every odd cycle of length greater than 5 has a chord**
  (not hard to check, this is because that $G\setminus\{ij\}$ is bipartite, therefore
  - every odd cycle must **contain the edge** $ij$
  - every odd cycle must contain **two of neighbor vertices of** $j$
    (two of $\{g, h, k_1, k_2, k_3\}$)
Concluding Remark

- Robertson’s conjecture is true for many cases
  - cubic graphs
  - if there is a girth cycle (of length 5) $C$ whose neighbor contains a path of length 2
  - also the neighbor of any girth cycle cannot be independent

Thus the cases that Robertson’s conjecture is not true are in a very specific situation which is that

For any girth cycle $C$, the neighbor $N(C)$ consists of independent edges (matchings) and isolated vertices. Therefore the candidates for counterexamples to Robertson’s conjecture are narrowed down to graphs with very specific structures.
Our counterexample contains the Petersen graph with two matching edges of distance 3 subdivided
(the base graph $G_0$ plus two paths joining two pairs of diagonal vertices)
Concluding Remark

- there are many such counterexamples, but they should have some substructure in common with our counterexample

(modifying Robertson’s conjecture)

**Conjecture 1**: Let $G$ satisfy those three conditions, and $C$ be any 5-cycle Then $G$ is either the Petersen Graph or contains a subgraph that contains $C$ and is isomorphic to $G_0$. 

![Diagram of the Petersen Graph]
Concluding Remark

- **Conjecture 2**: Let $G$ satisfy those three conditions and $C$ be any 5-cycle of $G$. Then $G$ is either the Petersen Graph or contains a subgraph that contains $C$ and is isomorphic to the Petersen graph with two matching edges of distance 3 subdivided.

- Both conjectures claim that $G$ tries to be the Petersen Graph around every girth cycle $C$. 
Concluding Remark

- $G$ (with these three conditions) is not a perfect graph (it has an odd hole). But it only has the odd hole of size 5.
- Our counterexample is almost a bipartite graph (bipartite if remove the edge $ij$), and the bipartite graphs are perfect graphs.
- any structure result for this class of graph?
- how ”close” are these graphs to perfect graphs?
- any coloring result related to this type of graphs?
Thank You!