Planar hypohamiltonian graphs

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A graph $G$ is called hypohamiltonian, if $G$ is non-hamiltonian, but $G - v$ is hamiltonian for any $v \in V(G)$

Starting point of the study of hypohamiltonian graphs: Sousselier, 1963

Smallest hypohamiltonian graph: the Petersen graph (Kempe, 1886)

In 1966, Gallai asked whether any connected graph possesses a vertex which lies on all longest paths

Motivated by above question, denote by $C_{j}^{k} (P_{j}^{k}) \ [C_{j}^{k} (P_{j}^{k})]$ the smallest order of a [planar] $k$-connected graph in which any $j$ vertices are avoided by some longest cycle (path)
Definitions and History

- $C_1^j = C_1^j = 3j + 3$ (sharp!) are given by the following simple construction (Thomassen, 1976)

- $C_2^1 = 10$ and $C_3^1 = 10$ due to Petersen’s graph
- Nothing is known for $C_4^1$
\[ C^1_2 = 15, \text{ Thomassen (1976)} \]
\[ \overline{C_3^1} \leq 124, \text{ Grünbaum (1974)} \]

- Each vertex is missed by a cycle of length \( 121 = n - 3 \)
Due to a theorem of Tutte from 1956 we know that for all $j$

$$C^j_4 = C^j_5 = \infty$$

Chvátal asked in 1972 whether there exist planar hypohamiltonian graphs.

Grünbaum conjectured their nonexistence.

Thomassen proved in 1976 that there are infinitely many.

His smallest example has order 105, so $C^1_3 \leq 105$
Theorem (Hatzel, 1979)

There exists a planar hypohamiltonian graph on 57 vertices, and we have $C_3^1 \leq 57$, $P_3^1 \leq 224$, $C_3^2 \leq 6758$ and $P_3^2 \leq 26378$. 
Theorem (CTZ and T. Zamfirescu, 2007)

There exists a planar hypohamiltonian graph on 48 vertices, and we have $C_3^1 \leq 48$, $P_3^1 \leq 188$, $C_3^2 \leq 4277$ and $P_3^2 \leq 16926$. 
Theorem (Araya and Wiener, 2011)

There exists a planar hypohamiltonian graph on 42 vertices, and we have $C^1_3 \leq 42$, $P^1_3 \leq 164$, $C^2_3 \leq 3701$ and $P^2_3 \leq 14694$. 
Theorem (Jooyandeh, McKay, Östergård, Pettersson, and CTZ)

There exists a planar hypohamiltonian graph on 40 vertices, and we have $C_3^1 \leq 40$, $P_3^1 \leq 156$, $C_3^2 \leq 2625$ and $P_3^2 \leq 10350$. 
The Araya-Wiener Theorem

**Theorem (Araya and Wiener, 2011)**

*There exists a planar hypohamiltonian graph on \( n \) vertices for every \( n \geq 76 \).*

**Theorem (CTZ)**

*There exists a planar hypohamiltonian graph on \( n \) vertices for every \( n \geq 48 \).*
Sketch of the proof:

- Let $G$ be a graph with a 4-cycle $(v_1, v_2, v_3, v_4) = C$. Define $\text{Th}(G^C)$ as the graph obtained from $G$ by deleting the edges $(v_1, v_4)$ and $(v_2, v_3)$ and adding a new 4-cycle $(v'_1, v'_2, v'_3, v'_4)$ and the edges $(v_i, v'_i)$ to $G$.

- This operation was introduced by Thomassen in 1981, where he (essentially) proved the following:

- Let $G$ be a planar non-hamiltonian graph having a 4-cycle $(a, b, c, d) = C$. Then $\text{Th}(G^C)$ is also a planar non-hamiltonian graph.

- Let $G$ be a planar hypohamiltonian graph having a 4-cycle $(a, b, c, d) = C$ and suppose that the vertices $a, b, c, d$ have degree 3. Then $\text{Th}(G^C)$ is also a planar hypohamiltonian graph.
Exemplifying the application of the Thomassen operation Th
In the following we can put $\text{Th}(G^C) = \text{Th}(G)$, as $C$ will be unique.

Consider the following planar hypohamiltonian graphs: the Araya-Wiener graph $\Gamma$, the new 45-vertex graph $\Lambda_{45}$, the 48-vertex graph $Z$, and the new 51-vertex graph $\Lambda_{51}$.

Prove that $\text{Th}(\Gamma), \text{Th}(\Lambda_{45}), \text{Th}(Z), \text{and } \text{Th}(\Lambda_{51})$ are planar hypohamiltonian graphs.

Use the Thomassen operation:

- $\Gamma \leadsto 42 + 4k$
- $\Lambda_{45} \leadsto 45 + 4k$
- $Z \leadsto 48 + 4p$
- $\Lambda_{51} \leadsto 51 + 4k$
The Thomassen Operation
Grinbergian Graphs
Cubic Case and Crossing Number

Proof

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Planar hypohamiltonian graphs
Theorem (Grinberg, 1968)

Given a planar graph $G$, a hamiltonian cycle $C$ in $G$, and $f_i$ ($f'_i$) $i$-gons inside (outside) of $C$, we have

$$\sum_i (i - 2)(f_i - f'_i) = 0.$$
Consider

\[ I_j = \{ i \in \mathbb{N} : i \geq 3 \text{ and } i = j \text{ mod. } 3 \}, \]

and let \( P_j \) be the family of all polygons for which their number of vertices lies in \( I_j \). We call a graph Grinbergian, if it is planar and of one of the following four types.

**Type 1.** Every face but one belongs to \( P_2 \).

**Type 2.** Every face belongs to \( P_2 \), except for five quadrilaterals, one of which (i) is adjacent to the other four, and (ii) has two non-adjacent cubic vertices.

**Type 3.** Every face belongs to \( P_2 \), except for three faces which share a common cubic vertex and belong to the same \( P_j \), \( j \in \{0, 1\} \).

**Type 4.** Every face is a quadrilateral, and \( f \) is odd.
Using Grinberg’s Criterion it is easily seen that every Grinbergian graph is non-hamiltonian.

Every Grinbergian hypohamiltonian graph is either of Type 1, in which case its exceptional face belongs to $P_1$, or of Type 2.

**Theorem (Jooyandeh, McKay, Östergård, Pettersson, and CTZ)**

*There exist exactly seven hypohamiltonian graphs of Type 1 which have order 42, and none on fewer vertices.*
The smallest known planar **cubic** hypohamiltonian graph has 70 vertices.

**Theorem (Araya and Wiener, 2011)**

*There exist planar cubic hypohamiltonian graphs of order \( n \) for every even \( n \geq 86 \).*

**Theorem (CTZ)**

*There exist planar cubic hypohamiltonian graphs of order \( n \) for every even \( n \geq 74 \).*
Theorem (CTZ)

For every $k \geq 0$ there exists an $n_0(k)$ such that for all $n \geq n_0$ there exists a hypohamiltonian graph which has order $n$ and crossing number $k$.

Closing Remarks

- There exist infinitely many planar hypohamiltonian graphs with trivial automorphism group
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Closing Remarks

- There exist infinitely many planar hypohamiltonian graphs with trivial automorphism group.
- By a Theorem of Thomassen planar hypohamiltonian graphs have minimum degree 3. Denote by $p_k$ the order of the smallest planar hypohamiltonian graph $G$ with maximum degree $k$. We have $44 \leq p_3 \leq 70$, $18 \leq p_4 \leq 40$, $18 \leq p_5 \leq 40$ and $18 \leq p_6 \leq 42$. 

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