Proper Trees and Contractible Hamiltonian Cycles in Combinatorial 2-Manifolds

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Outline

1. Combinatorial 2-Manifolds
   - Examples
   - Dual

2. Proper Tree: Triangulated Surfaces
   - Properties
   - Results

3. Proper Tree: Polyhedral Maps
   - Properties
   - Results

4. A Remark

5. Future Work
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Preliminary Terms

- **Simplicial Complex**: Let $V$ be a finite set. A finite collection $K$ of subsets of $V$ is called a simplicial complex if the subset of a member of $K$ is again a member. Each subset of $K$ is called a face. We usually identify a 2-dimensional simplicial complex with the set of faces in it.

- **Geometric carrier**: Let $X$ be a finite simplicial complex and $V(X) = \{v_1, \ldots, v_n\}$. Choose a set of $n$ points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^N$ (for some sufficiently large $N$) in such a way that a subset $S = \{x_{j_1}, \ldots, x_{j_{i+1}}\}$ of $i + 1$ points is affinely independent if $\sigma = \{v_{j_1}, \ldots, v_{j_{i+1}}\}$ is a simplex of $X$. The convex set spanned by $S$ is called the geometric simplex corresponding to $\sigma$ and denoted by $|\sigma|$. 
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Since $V(X)$ is finite we can choose $N$ so that $\sigma \cap \gamma = \emptyset$ implies $|\sigma| \cap |\gamma| = \emptyset$. The set
\[ X := \{ |\sigma| : \sigma \in X, \sigma \cap \gamma = \emptyset \Rightarrow |\sigma| \cap |\gamma| = \emptyset \} \]

is called a geometric simplicial complex corresponding to $X$ or a geometric realization of $X$. The topological space $|X| := \cup_{\sigma \in X} |\sigma|$ is called a geometric carrier of $X$. 

\[ K := \{\{a\}, \{u\}, \{v\}, \{w\}\} \]
\[ \cup \{\{au\}, \{aw\}, \{uv\}, \{uw\}, \{vw\}\} \]
\[ \cup \{\{auw\}, \{uvw\}\} \]
Link of a vertex: If $v$ is a vertex of a simplicial complex $X$, then the link of $v$ in $X$, denoted by $\text{lk}_X(v)$ (or $\text{lk}(v)$), is the simplicial complex $\{ \tau \in X : v \notin \tau, \{v\} \cup \tau \in X \}$.

Combinatorial 2-manifold: A finite 2-dimensional simplicial complex $K$ is called a combinatorial 2-manifold if $|K|$, i.e. the geometric carrier of $K$ is a topological 2-manifold. It is clear that a finite simplicial complex $K$ is a combinatorial 2-manifold if and only if link of each vertex $\text{lk}_K(v)$ in $K$ is a cycle for each vertex $v$ of $K$. 
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Example

Four degree-regular combinatorial 2-manifolds of positive Euler characteristic.
Example

Some degree regular orientable combinatorial 2-manifolds of Euler characteristic $0$, $n \geq 7$. 

$$
\begin{array}{cccccc}
3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\quad \cdots \quad 
\begin{array}{cccc}
1 & 2 & 3 \\
n & 1 & 2 & 3 \\
\end{array}
$$

$T_{n,1,2}$

$T_{n-2,n-1,n}$

$T_{6,2,2}$

$T_{4,4,2}$
Example

Some degree regular non-orientable combinatorial 2-manifolds of Euler characteristic 0.
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An orientable degree-regular combinatorial 2-manifolds of Euler characteristic $-2$ on vertex set $\{0, 1, \ldots, 11\}$. 
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In the dual of the triangulation (which is a $\{7, 3\}$ - equivelar map on same surface) the corresponding object is a tree, and in fact it is a peculiar tree which we later named as Proper tree.

D. Barnette (UCDavis): “it looks to me like such a tree might be a necessary and sufficient condition for a contractable Hamiltonian circuit. Can the condition “equiveler” be relaxed? I know it’s easy to get triangulation’s without Hamiltonian circuits by capping faces, but this introduces lots of three valent vertices. What if you put a lower limit on the valence of vertices?”

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1. whenever two vertices $u_1$ and $u_2$ of $T$ belong to a face $F$ (in fact they lie on the boundary cycle of $F$), a path $P[u_1u_2]$ joining $u_1$ and $u_2$ in boundary of $F$ belongs to $T$, and
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Definition & Example

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Properties of Proper Tree

- Let $v \in V(T)$ be a vertex in a proper tree $T$. Then $\deg(v) \leq 3$.

- Let $T$ be a proper tree and $m$ be the number of vertices of degree 3 in $T$. Then the number of vertices of degree one in $T = m + 2$.

- Let $T$ be a proper tree in a polyhedral map $M$ of type $\{q, 3\}$ on a surface $S$. Then $T \cap F \neq \emptyset$ for any face $F$ of $M$.

- Let $K$ be a $n$ vertex equivelar triangulation of a surface $S$. Let $M$ denote the dual map corresponding to $K$ and $T$ be a $n - 2$ vertex proper tree in $M$. Let $D$ denote the subcomplex of $K$ which is dual of $T$. Then $D$ is a triangulated 2-disk and the boundary of $D$ $\bd(D)$ is a Hamiltonian cycle in $K$. 
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**Theorem:** Let $S$ denote a surface which has an equivelar triangulation $K$. The edge graph $EG(K)$ of $K$ has a contractible Hamiltonian cycle if and only if the edge graph of dual map $M$ corresponding to $K$ has a proper tree.

**Map & Polyhedral Map:** A Map on a surface $S$ is an embedding of a finite simple graph $G$ such that the closure of components of $S \setminus G$ is $p$-gonal 2-disc for $p \geq 3$. The components are also called facets.

The map $M$ is called a Polyhedral Map if non-empty intersection of any two facets of the map is either a vertex or an edge. The $\{p, q\}$ - equivelar maps are examples of polyhedral maps.
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Jointly with D. Maity

**Definition:** Consider a polyhedral map $K$ on a surface $S$ that has $n$ vertices. Let $M$ denote the dual map of $K$ and $T := (V, E)$ denote a tree in the edge graph $EG(M)$ of $M$. We say that $T$ is a proper tree if the following conditions hold:

1. \[ \sum_{i=1}^{k} \deg(v_i) = n + 2(k - 1), \] where $V = \{v_1, v_2, ..., v_k\}$ and $\deg(v)$ denotes degree of $v$ in $EG(M)$

2. Whenever two vertices $u_1$ and $u_2$ of $T$ lie on a face $F$ in $M$, a path $P[u_1, u_2]$ joining $u_1$ and $u_2$ in the boundary $\partial F$ of $F$ is a subtree of $T$, and

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Properties

- Let $T$ be a proper tree in a general polyhedral map $M$ on a surface $S$. Then $T \cap F \neq \emptyset$ for any facet $F$ of $M$.

- Let $K$ be a $n$ vertex polyhedral map and $M$ denote the dual polyhedron corresponding to $K$. Let $T$ be a $k$ vertex proper tree in $M$. If $D$ denotes the subcomplex of $K$ which is dual of $T$ then $D$ is a 2-disk and the boundary $\partial D$ of $D$ is a Hamiltonian cycle in $EG(K)$.

- The results are also true, in particular, in case of equivelar maps.
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Results

- **Theorem:** The edge graph $EG(K)$ of a map $K$ on a surface has a contractible Hamiltonian cycle if and only if the edge graph of corresponding dual map of $K$ has a proper tree. □

- Let $M$ denote the dual map of a $n$ vertex $\{p, q\}$-equivelar map $K$ on a surface $S$. If $\frac{n-2}{p-2} = m$ is an integer then $M$ has an admissible proper tree on $m$ vertices.
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- **Remark:** Observe that if the map $K$ is \{p, q\}-equivelar then \( k = \frac{n - 2}{p - 2} \). Thus, for an equivelar triangulation on \( n \) vertices the proper tree has exactly \( n - 2 \) vertices.

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Where does this lead to

- In a 2005 paper Mori and Nakamoto have shown that any two triangulations on the projective plane with \( n \) vertices can be transformed into each other by at most \( 8n-26 \) diagonal flips, up to isotopy. To prove it, we focus on triangulations on the projective plane with contractible Hamilton cycles.

- In particular they have proved that \( G \) and \( H \) be two triangulations on the projective plane with \( n \) vertices, each of which has a contractible Hamilton cycle. Then \( G \) and \( H \) can be transformed into each other by at most \( 6n - 12 \) diagonal flips, preserving their Hamilton cycles.

- This will be one of the directions we will be looking at.

- Another obvious question will be looking for similar results about non-contractible Hamiltonian cycles.
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Questions, Comments, Suggestions!!

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