Arc-disjoint paths with prescribed endvertices in generalizations of tournaments

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Definition

Given $D = (V, A)$ and (not necessarily distinct) vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. A **weak k-linkage** from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$ is a collection of $k$ arc-disjoint routes $P_1, \ldots, P_k$ such that $P_i$ is an $(s_i, t_i)$-path (or a proper cycle containing $s_i = t_i$) for each $i \in [k]$. 
Complexity of the general problem

**Problem**

WEAK $k$-LINKAGE PROBLEM: Given $D = (V, A)$ and not necessarily distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$; Does $D$ have a weak $k$-linkage from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$?

Theorem (Fortune, Hopcroft and Wyllie, 1980)
The weak $k$-linkage problem is NP-complete for $k \geq 2$.

Theorem (Shiloach, 1979)
Every $k$-arc-strong digraph ($d_X + (X) \geq k \forall \emptyset \subset X \subset V$) has a weak $k$-linkage for every choice of $k$ pairs of terminals.
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Every $k$-arc-strong digraph ($d^+(X) \geq k \ \forall \emptyset \subset X \subset V$) has a weak $k$-linkage for every choice of $k$ pairs of terminals.
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Theorem (Fradkin-Seymour, 2011)

For every fixed $\alpha$, $k$ the weak $k$-linkage problem is polynomial for digraphs with independence number at most $\alpha$. 

\( D = (V, A) \). \( O = v_1, \ldots, v_n \) ordering of \( V \) has **cutwidth** \( \theta \) if for every \( j = 1, \ldots, n - 1 \)

\[ v_j + 1 \leq \theta \]

\( D \) has cutwidth \( \theta \) if \( \exists \) \( O \) with cutwidth \( \theta \)
Theorem (Fradkin-Seymour, 2011)

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Corollary

For every fixed $\theta$, $k$ the weak $k$-linkage problem is polynomial, for digraphs with at most $\theta$ directed cycles.
$H$ an induced subdigraph of $D$ is a module if

$$\forall a, b \in V(H), \; v \in V(D \setminus H) \; \mu(va) = \mu(vb), \; \mu(av) = \mu(bv).$$

(If $D$ is simple, we simply say that every vertex of $H$ must have the same in and out neighborhood)
$D$ is decomposable if $\exists$ partition of $V$ into modules $H_1, \ldots H_s$, $s \geq 2$. We write $D = S[H_1, \ldots, H_s]$, where $S$ is the adjacency (or quotient) digraph of $H_1, \ldots H_s$. 

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\[ S = \begin{array}{c}
H_1 \\
\rightarrow \\
H_2 \\
\rightarrow \\
H_3
\end{array} \]
**Decomposable digraphs**

A digraph $D$ is decomposable if $\exists$ partition of $V$ into modules $H_1, \ldots, H_s$, $s \geq 2$. We write $D = S[H_1, \ldots, H_s]$, where $S$ is the adjacency (or quotient) digraph of $H_1, \ldots, H_s$.

![Diagram](image)

$\Phi$ class of digraphs. $D$ is totally $\Phi$-decomposable if either $D \in \Phi$ or $D = S[H_1, \ldots, H_s]$, with $S \in \Phi$ and $H_i$ totally $\Phi$-decomposable, $i = 1, \ldots, s$.

The digraph in the figure is totally $\Phi$-decomposable with $\Phi = P_3 \cup C_3 \cup P_1$.
We say that a class of digraph $\Phi$ is bombproof if

- $\exists$ poly algorithm to find a total $\Phi$-decomposition
- For every fixed $k$, $c$, weak $k$-linkage is poly on

  $\Phi(c) := \{ D' \mid \exists D \in \Phi \ D' \text{ is obtained from } D \text{ blowing up } \leq c \text{ vertices to digraphs of size } \leq c \}$
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If $\Phi = \{P_3\}$, then

is in $\Phi(3)$
Main theorem

Let $\Phi$ be a bombproof class of digraphs. For every fixed $k$ the weak $k$-linkage problem is polynomial for totally $\Phi$-decomposable digraphs.
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Lemma

A YES instance has always a $k$-weak linkage using
- At most $2k$ arcs inside any module
- No arc inside any module without terminals
Algorithm

Given $D$ totally $\Phi$-decomposable, $\Pi$ list of terminals and $F$ set of arcs of bounded size, decides whether $D \setminus F$ has a weak-linkage for $\Pi$.

1. Find a total $\Phi$-decomposition $D = S[H_1, \ldots, H_s]$. If $\Pi = \emptyset$ output YES. If $D \in \Phi$ decide the problem with a poly algorithm on $\langle D \setminus F, \Pi \rangle$.

2. Find the modules $K_1, \ldots, K_l$ containing terminals.
(3) For every partition $\Pi = \Pi_i \cup \Pi_e$ and for every choice of sets of $\leq 2k$ arcs $F_1 \subset A(K_1), \ldots, F_l \subset A(K_l)$
   - Run recursively the algorithm on $\langle K_1, F \cup F_1, \Pi_i \cap K_1 \rangle, \ldots, \langle K_l, F \cup F_l, \Pi_i \cap K_l \rangle$. If they are all YES
     - Blow up the vertices of $S$ corresponding to $K_1, \ldots, K_l$ to the digraphs formed by $F_1, \ldots, F_l$. Run a poly algorithm on $\langle S(\text{blown}) \setminus F, \Pi_e \rangle$. If it is YES, output YES

(4) Output NO
$D = (V, A)$ is **quasi-transitive** if $xy, yz \in A$ implies that $zx \in A$ or $xz \in A$.

**Theorem (Bang-Jensen and Huang)**

Let $D$ be a digraph which is quasi-transitive.

- If $D$ is not strong, then $D = T[H_1, \ldots, H_t]$, where $T$ is a transitive oriented graph ($\Rightarrow$ acyclic) and $H_1, \ldots, H_t$ are strong quasi-transitive.

- If $D$ is strong, then $D = S[Q_1, Q_2, \ldots, Q_s]$, where $S$ is strong semicomplete and $Q_1, \ldots, Q_s$ are either single vertices or non-strong quasi-transitive.
The class $\Phi_1$

$\Phi_1 := \text{Semicomplete digraphs} \cup \text{Acyclic digraphs}$

By the previous characterization quasi-transitive digraphs are totally $\Phi_1$-decomposable.
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**Lemma**

$\Phi_1$ is bombproof.

**Proof sketch**

*A poly algorithm for total $\Phi_1$-decomposition was given by B-J and Gutin.*

*Given $D \in \Phi_1$*

- *If $D$ is semicomplete, then $D(\text{blown})$ misses $O(c^3)$ arcs to be semicomplete.*
- *If $D$ is acyclic, then $D(\text{blown})$ has $O(c \cdot (ck)^c)$ cycles.*
Corollary 1

For every fixed $k$ there exists a polynomial algorithm for the weak $k$-linkage problem for quasi-transitive digraphs.
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$D$ is extended semicomplete if $D = S[H_1, \ldots, H_s]$, where $S$ is semicomplete and $H_1, \ldots, H_s$ are independent sets.

Corollary 2

For every fixed $k$ there exists a polynomial algorithm for the weak $k$-linkage problem for extended semicomplete digraphs.
\( D \) is round if we can label its vertices \( v_1, \ldots, v_n \) so that \( \forall \ i, \)
\[
N^+(v_i) = \{ v_{i+1}, \ldots, v_{i+d^+(i)} \} \quad \text{and} \quad N^-(v_i) = \{ v_{i-d^-(i)}, \ldots, v_{i-1} \}.
\]
$D$ is round if we can label its vertices $v_1, \ldots, v_n$ so that $\forall \ i$, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(i)}\}$ and $N^-(v_i) = \{v_{i-d^-(i)}, \ldots, v_{i-1}\}$.

$D$ is round decomposable if $D = R[H_1, \ldots, H_r]$, where $R$ is a round digraph and $H_1, \ldots, H_r$ are semicomplete digraphs.
The class $\Phi_2$

$\Phi_2 := \text{Round digraphs } \cup \text{Semicomplete digraphs}$

**Lemma**

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**Crux**

$D$ round and distinct terminals. If all reverse round enumerations have $\theta \geq 36k^3$, then YES.
Partition into consecutive intervals of size $2k$.

For $i = 1, ..., k$ color $s_i, t_i$ with color $i$. Color at least one vertex of color $i$ in all intervals.

From $s_i$ go to the next $i$-vertex (if any) otherwise go to the furthest vertex (which has high $d^+$ and sees an $i$-vertex).

The paths only share high out-degree vertices $\Rightarrow$ they are arc-disjoint.

**Corollary**

*For every fixed $k$ there exists a polynomial algorithm for the weak $k$-linkage problem for round decomposable digraphs.*
D is **locally semicomplete** if $\forall x \in V(D)$, $N^+(x)$ and $N^-(x)$ induce semicomplete digraphs.

**Theorem**

A connected LSD is either
- *round decomposable*, or
- has independence number $\leq 2$. 

Using Fradkin-Seymour algorithm we get

**Corollary**

For every fixed $k$ there exists a polynomial algorithm for the *weak $k$-linkage problem* for LSD.
Locally semicomplete digraphs (LSD)

A connected LSD is either
- round decomposable, or
- has independence number \( \leq 2 \).

Using Fradkin-Seymour algorithm we get

Corollary

For every fixed \( k \) there exists a polynomial algorithm for the weak \( k \)-linkage problem for LSD.