Index Theory on Singular Manifolds: a Groupoids’ approach

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In Noncommutative Geometry: pseudodifferential calculus ↔ groupoid

**Background**: index theorem on foliations (Connes, Skandalis)

**Problem**: apply Connes’ approach to singular manifolds: understand the index theory on singular manifolds in terms of operators algebras, using groupoids methods.

**Scheme**: to define a pseudodifferential calculus, define a groupoid and use the general tools developed for the pseudodifferential calculus on a groupoid.

**Collaborators**: R. Lauter, P.Y. Le Gall, V. Nistor, F. Pierrot
Definition 1  Groupoid : small category in which all morphisms are invertible

\[ s(\gamma') = r(\gamma) \]

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\[ r(\gamma') \]

\[ \gamma' \circ \gamma \]
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Additional structure: topology, smooth structure (Lie groupoid), continuous family groupoid (A. Patterson)
3 – Pseudodifferential calculus on groupoids

- $G$ is a Lie groupoid (more generally a continuous family groupoid) $\rightsquigarrow$ algebra of pseudodifferential operators $\Psi^\infty(G)$
- Pseudodifferential operator on $G$: $G$-equivariant continuous family of pseudodifferential operators on the fibers of $G$
- The structure we need: smooth fibers, continuity on the basis. Continuous family groupoids generalize the holonomy groupoid of a $C^{\infty,0}$-foliation (A. Connes)
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**Examples** :
- If \( M \) is a manifold without boundary, and \( G = M \times M \), \( \Psi^\infty(G) \) is the algebra of pseudodifferential operators on \( M \).
- If \( G \) is a Lie group, \( \Psi^\infty(G) \) is the algebra of \( G \)-equivariant pseudodifferential operators on \( G \).
- If \( M \) is a manifold with corners, there exists a groupoid such that \( \Psi^\infty(G) \) is the \( b \)-calculus of Melrose.
  Can be extended to manifolds in which the corners are not embedded.
**Atiyah-Singer exact sequence**

\[ 0 \to C^\ast(G) \to \Psi^0(G) \overset{\sigma}{\to} C(S^\ast(G)) \to 0 \]

\((S^\ast(G) : \text{cosphere bundle of the Lie algebroid } A(G))\)

**Theorem 1** *The analytic index*

\[ \text{Ind}_a : K^0(A^\ast(G)) \to K_0(C^\ast(G)) \]

*is induced by the tangent groupoid* \(G \times ]0, 1] \cup A(G) \times \{0\}.\)
Restriction morphism: if $Y \subset G^{(0)}$ is closed and invariant, 
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Fredholmness

**Theorem 2** $G$ groupoid such that $\exists U \subset G^{(0)}$, $G_U = U \times U$; let $Y = G^{(0)} \setminus U$. Then an operator of order 0 is Fredholm if and only if its symbol $\sigma(P)$ is invertible as well as $R_Y(P)$.

This gives a general condition of “total ellipticity”.
Proof:

\[
\begin{array}{cccccc}
0 & \rightarrow & C^*(\mathcal{G}) & \rightarrow & \overline{\Psi^0}(\mathcal{G}) & \xrightarrow{\sigma^0} & C(S^*\mathcal{G}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C^*(\mathcal{G}_Y) & \rightarrow & \overline{\Psi^0}(\mathcal{G}_Y) & \rightarrow & C(S^*\mathcal{G}_Y) & \rightarrow & 0 \\
\end{array}
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0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

\[
0 \rightarrow \mathcal{K} \rightarrow \overline{\Psi}^0(\mathcal{G}) \rightarrow C(S^*\mathcal{G}) \times_f \overline{\Psi}^0(\mathcal{G}_Y) \rightarrow 0
\]


\textbf{Spectral invariance}

length function of polynomial growth on $G \xrightarrow{\sim} \text{Schwartz space on } G$, $S(G)$.

\textbf{Theorem 3} $S(G)$ is a subalgebra of $C^*(G)$, closed under holomorphic functional calculus. Same with $\Psi^0(G') + S(G')$ in $\overline{\Psi^0(G')}$. 


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This implies that if an operator of \( \Psi^0(G) + S(G) \) is Fredholm, it has a parametrix in \( \Psi^0(G) + S(G) \).
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**Theorem 3** $\mathcal{S}(G)$ is a subalgebra of $C^\ast(G)$, closed under holomorphic functional calculus. Same with $\Psi^0(G') + \mathcal{S}(G)$ in $\overline{\Psi^0(G')}$. 

This implies that if an operator of $\Psi^0(G') + \mathcal{S}(G)$ is Fredholm, it has a parametrix in $\Psi^0(G') + \mathcal{S}(G)$.

We also defined other spectrally invariant algebras in the case of the groupoid of the cusp-calculus.
5 – Application to manifolds with corners

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"Universal" groupoid:

\[ \mathcal{G}(X) = \left\{ (x, y, \alpha), x, y \in X, \text{codim}(x) = \text{codim}(y), \alpha : N_y F(y) \xrightarrow{\sim} N_x F(x) \right\}. \]

If \( X \) and \( X' \) have the same codimension, the analytic indices take their values in the same group, known.
Application to manifolds with corners:

"$b$"-groupoid (for manifold with embedded corners):

\[ \Gamma_b(X) = \bigcup_{F \text{ face of } X} F \times F \times \mathbb{R}^{\text{codim}F} \]

\[ = \{(x, y, \lambda_1, \ldots, \lambda_k) \in X \times X \times \mathbb{R}_+^k, \rho_i(x) = \lambda_i \rho_i(y) \}\]  

$\rho_i$: defining functions of the faces
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\( \rho_i \) : defining functions of the faces

\( \Gamma_b(X) \) is an open subgroupoid of \( \mathcal{G}(X) \)

For the cusp-calculus, homeomorphic groupoid

Length function: \( \phi(x, y, \lambda_1, \ldots, \lambda_k) = \| \log(\lambda_i) \| \)